(i) If $c \geq 0$, $\int cf\,d\mu = c\int f\,d\mu$.

(ii) If $f \leq g$, then $\int f\,d\mu \leq \int g\,d\mu$.

Proof

Homework

To establish further properties, and indeed to compute integrals, we have to establish a fundamental convergence result. Recall that the Riemann integral is practical because we can compute it to arbitrary precision by using a specified sequence of approximations. – start –

Theorem 5.1.5 Montone Convergence Theorem

If $\{f_n\}$ is a sequence in L^+ with $f_n \leq f_{n+1}$ for all n and $f = \lim_n f_n$ $(= \sup_n f_n)$, then $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$.

This result gives conditions that allow us to exchange $\int$ with a limit. Recall our discussion about Riemann integration and limits.

This shows that we can compute $\int f\,d\mu$ for $f \in L^+$ by looking at $\int S_n\,d\mu$ for a sequence of simple functions $\{S_n\}$ with $S_n \uparrow f$.

<u>Proof</u>

$\{\int f_n\,d\mu\}$ is an increasing sequence of numbers, so its limit exists (which might be ∞). Moreover, $\int f_n\,d\mu \le \int f\,d\mu$ for all n, so $\lim_n \int f_n\,d\mu \le \int f\,d\mu$. For the reverse, fix $0 < \alpha < 1$ and let φ be a simple function with $0 \le \varphi \le f$, then define

$$E_n = \{x : f_n(x) \ge \alpha \varphi(x)\}.$$

$\{E_n\}$ is an increasing sequence ($E_n \subset E_{n+1}$) of measurable sets whose union is X

We have

$$\int f_n \, d\mu \ge \int_{E_n} f_n \, d\mu \ge \alpha \int_{E_n} \varphi \, d\mu.$$

By Theorem 5.1.2 (iv) and (3.2.6),

$\lim_n \int_{E_n} \varphi \, d\mu = \int \varphi \, d\mu$, so $\lim \int f_n \, d\mu \ge \alpha \int \varphi \, d\mu$.

Since this holds for $0 < \alpha < 1$, $\lim_n \int f_n \, d\mu \ge \int \varphi \, d\mu$. Taking the sup over all simple functions $\varphi \le f$ gives the result.

<u>Example 5.1.5</u>

We compute $\int_{[1,\infty)} \frac{1}{x^2} \, d\mu_L$.

Define
$$S_1(x) = \begin{cases} 1, & x=1 \\ \frac{1}{2^2}, & 1<x<2 \\ 0, & 2<x \end{cases}$$

$$\int_{(0,\infty)} S_1 \, d\mu_L = 1 \cdot \frac{1}{2^2}.$$

$$S_2(x) = \begin{cases} 1, & x=1 \\ 1/(3/2)^2, & \frac{2}{2} < x \le 3/2 \\ 1/(4/2)^2, & 3/2 < x \le 4/2 \\ 1/(5/2)^2, & 4/2 < x \le 5/2 \\ 1/(6/2)^2, & 5/2 < x \le 6/2, \\ 0, & 3 < x. \end{cases}$$

$$\int_{[1,\infty)} S_2(x) \, d\mu_L = 2\left[\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2}\right]$$

In general, set

$$S_n(x) = \begin{cases} 1, & x=1, \\ 1/(1+\frac{k}{2^{n-1}})^2, & 1+\frac{k-1}{2^{n-1}} < x \le 1 + \frac{k}{2^{n-1}}, \quad 1 \le k \le n2^n \\ 0 & n+1 < x. \end{cases}$$

Show $0 \le S_n \le S_{n+1}$ and $\lim_{n\to\infty} S_n = f$ on $[1,\infty)$. Since

$$\frac{1}{(k+1)k} < \frac{1}{k \cdot k} < \frac{1}{(k-1)\cdot k}$$

it is possible to show

$$2^{n-1}\left(\frac{1}{(2^{n-1}+2)(2^{n-1}+1)} + \dots + \frac{1}{((n+1)2^{n-1}+1)(n+1)2^{n-1}}\right)$$

$$< \int_{(1,\infty)} S_n \, d\mu_L$$

$$< 2^{n-1}\left(\frac{1}{2^{n-1}(2^{n-1}+1)} + \dots + \frac{1}{((n+1)2^{n-1}-1)(n+1)2^{n-1}}\right)$$

or

$$\frac{1}{1+\frac{1}{2^{n-1}}} - \frac{1}{n+1+\frac{1}{2^{n-1}}} < \int_{(1,\infty)} S_n \, d\mu_L < 1 - \frac{1}{n}$$

Hence,

$$\lim_n \int_{(1,\infty)} S_n \, d\mu_L = 1 = \int_{[1,\infty)} \frac{1}{x^2} \, d\mu_L.$$

We can now prove some important properties of the integral.

<u>Theorem 5.1.6</u>

If $\{f_n\}$ is a finite or infinite sequence in L^+ and $f = \sum_n f_n$, then $\int f\,d\mu = \sum_n \int f_n\,d\mu$.

This extends Theorem 5.1.2 (ii). Notice we are talking about infinite sums however! Compare this to the Riemann integral.

<u>Proof</u>

First consider f_1 and f_2. By Thm 4.3.4, there are sequences of simple functions $\{\varphi_j\}$ and $\{\psi_j\}$ which are nonnegative and increase monotonically to f_1 and f_2 resp. $\{\varphi_j + \psi_j\}$ is a nonnegative sequence of simple functions that increases monotonically to $f_1 + f_2$, so Thm 5.1.5 implies

$$\int (f_1 + f_2)\,d\mu = \lim \int (\varphi_j + \psi_j)\,d\mu = \lim \int \varphi_j\,d\mu + \lim \int \psi_j\,d\mu$$
$$= \int f_1\,d\mu + \int f_2\,d\mu.$$

By induction, $\int \sum_{j=1}^{n} f_j \, d\mu = \sum_{j=1}^{n} \int f_j d\mu$. We let $n \to \infty$ and apply Theorem 5.1.5 again to obtain $\int \sum_{j=1}^{\infty} f_j d\mu = \sum_{j=1}^{\infty} \int f_j d\mu$.

Note: we used the montone convergence theorem twice!

We next prove some results pertaining to the integrand.

Theorem 5.1.7

If $f \in L^+$, then $\int f d\mu = 0$ if and only if $f = 0$ a.e.

Proof

If f is simple, i.e., $f = \sum_{j=1}^{n} a_j \chi_{E_j}$ where $a_j \geq 0$ and $\{E_j\}_{j=1}^{n} \subset \mathcal{M}$, then $\int f d\mu = 0$ if and only if for each j $a_j = 0$ and/or $\mu(E_j) = 0$.

In general, if $f = 0$ a.e. and φ is a simple function with $0 \le \varphi \le f$, then $\varphi = 0$ a.e., so $\int f\,d\mu = \sup_{\varphi \le f} \int \varphi\,d\mu = 0$. On the other hand,

$$\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$$

where

$$E_n = \left\{x : f(x) > \frac{1}{n}\right\}.$$

If it is false that $f = 0$ a.e., then $\mu(E_n) > 0$ for some n. Then, $f > n^{-1}\chi_{E_n}$, so

$$\int f\,d\mu \ge \int n^{-1}\chi_{E_n}\,d\mu = \frac{1}{n}\mu(E_n) > 0.$$

This is the first hint that behavior on sets of measure zero do not affect the integral. As a consequence, we get a generalization of Thm 5.1.5.

Theorem 5.1.8 (This is a good example of a test problem)

If $\{f_n\} \subset L^+$, $f \in L^+$, and $f_n(x)$ increases monotonically to $f(x)$ for a.e. x, then

$\int f \, d\mu = \lim_n \int f_n \, d\mu$.

Proof

Let $f_n(x) \uparrow f(x)$ for $x \in E$, where $\mu(E^c) = 0$.

Then $f = f \chi_E$ and $f_n = f_n \chi_E$ a.e., so

$$\int f \, d\mu = \int f \chi_E \, d\mu = \lim_n \int f_n \chi_E \, d\mu = \lim_n \int f_n \, d\mu$$

by Thm's 5.1.7 and 5.1.5.

The assumption that $\{f_n\}$ is montonically increasing a.e. is essential to the Montone Convergence Theorem (5.1.5).

Example 5.1.6 (Know!)

Consider $(\mathbb{R}, \mathcal{L}, \mu_L)$. $\{\chi_{(n,n+1)}\}$ is a sequence of measurable functions with

$$\chi_{(n,n+1)} \to 0 \quad \text{pointwise}$$

$$\int \chi_{(n,n+1)}\, d\mu_L = 1 \quad \text{for all } n.$$

However, the integral of the limit can never be larger than the limit of the integrals.

<u>Theorem 5.1.9</u> (Fatou's Lemma)

If $\{f_n\} \subset L^+$,

$$\int (\liminf f_n)\, d\mu \leq \liminf \int f_n\, d\mu_L.$$

Recall Thm 4.2.2.

<u>Proof</u>

For each k, $\inf_{n \geq k} f_n \leq f_j$ for $j \geq k$.

Hence, $\int \inf_{n\geq k} f_n\, d\mu \leq \int f_j\, d\mu$ for $j \geq k$. We let $k \to \infty$ and apply the M. C. T. (5.1.5)