large.

Book: how general should §7.4 be presented – or transition better

§7.4 Compact Sets in $C([a,b])$.

Example 7.2.1 shows that $C([a,b])$ is not compact.

Example 7.4.1

The sequence $\{x^n, n \in \mathbb{N}\}$ does not have a convergent subsequence in $C([a,b])$, i.e., there is no subsequence that converges uniformly.

– start –

This raises the issue of describing the compact subsets of $C([a,b])$.

By Theorem 3.2.1, if $K \subset C([a,b])$ is compact, then K is closed and bounded. Closed is rather obvious: K is closed if for any sequence of functions $\{f_n\}$ in K that converges to f in the metric of $C([a,b])$ (uniformly), we have $f \in K$.

Example 7.4.2 Let $F = \{f \in C([a,b]) \mid \sup_{a \le x \le b} |f(x)| < 1\}$.

F is not closed since, for example,

$$\left\{1-\frac{1}{n} \,\middle|\, n=1,2,3\ldots\right\}$$

is a sequence of functions in F that converges uniformly to $f(x)\equiv 1$, which is not in F.

Example 7.4.7

Let $F = \left\{ f \in C([a,b]) \,\middle|\, \sup_{a\le x\le b} |f(x)| \le 1 \right\}$.

We show F is closed. Choose a sequence $\{f_n\}$ in F that converges to f in $C([a,b])$. We show $f \in F$, that is $\sup_{a\le x\le b} |f(x)| \le 1$. Suppose that $\sup_{a\le x\le b} |f(x)| > 1$. There is an $\varepsilon > 0$ and an $x \in [a,b]$ such that $|f(x)| > 1+\varepsilon$. Because f is continuous, there is a $\delta > 0$ such that $|f(y)| > 1+\varepsilon/2$ for $y \in (x-\delta, x+\delta)\cap[a,b]$. But, for $y \in (x-\delta, x+\delta)\cap[a,b]$ and all n,

$$|f(y) - f_n(y)| \ge \big|\,|f(y)| - |f_n(y)|\,\big| \ge |1+\varepsilon/2 - 1| \ge \varepsilon/2,$$

which contradicts $\sup_{a\le y\le b} |f_n(y) - f(y)| \to 0$ as $n\to\infty$.

Back: explain equivalent language of uniform boundedness

$K \subset C([a,b])$ is bounded means there is a function $g \in C([a,b])$ and an M such that

$$d(f,g) = \sup_{a \le x \le b} |f(x) - g(x)| \le M$$

for all $f \in K$. Since such a g is itself bounded on $[a,b]$, we see there is an M such that

$$\sup_{a \le x \le b} |f(x)| \le M \quad \text{all } f \in K.$$

This motivates

<u>Definition 7.4.1</u>

Let (X,d) be a metric space, $A \subset X$, and F a set of functions from A into $\mathbb{R}^n$ with the usual metric. F is <u>uniformly bounded</u> on A if there is an M such that

$$\sup_{x \in A} \|f(x)\| \le M \quad \text{all } f \in F.$$

We have shown that if $F \subset C([a,b])$ is bounded, then F is uniformly bounded. The converse is obviously true.

Book: bother with uniform boundedness or just talk about bounded subsets?

Theorem 7.4.1

Let $F \subset C([a,b])$ be a set of continuous functions on $[a,b]$. Then, F is uniformly bounded on $[a,b]$ if and only if F is a bounded subset of $C([a,b])$.

Example 7.4.4

The set $F = \{x^n \mid n = 1, 2, 3\}$ is bounded on $[0,1]$ but not on $[0,2]$. The qualification of boundedness does depend on $[a,b]$. On the other hand, we do not expect that being merely closed and bounded guarantees compactness. In fact, $\{x^n \mid n = 1, 2, 3, \ldots\}$ is closed and bounded on $[0,1]$, but is not compact. We need something more.

Example 7.4.5

Book: idea, let's take a set constructed to be compact and derive a property that is characteristic.

Consider a sequence $\{g_n\}$ in $C([a,b])$ that converges in $C([a,b])$ to $g \in C([a,b])$. The set

$$K = \{g_n \mid n=1,2,3\ldots\} \cup \{g\}$$

is compact. If $\{f_n\}$ is a sequence in K, then if $\{f_n\}$ contains a function in K repeated infinitely often, then it has a subsequence that converges to an element of K. Otherwise, infinitely many of the functions $\{g_n\}$ are contained in $\{f_n\}$, and $\{f_n\}$ contains a subsequence that converges to g.

Since g is uniformly continuous on $[a,b]$, given $\varepsilon>0$, there is a δ_0 such that

$$|g(x)-g(y)| < \varepsilon/3 \text{ for } x,y \in [a,b],\ |x-y|<\delta_0.$$

Since $g_n \to g$ uniformly, there is an N such that

$$\sup_{a \le x \le b} |g(x)-g_n(x)| < \varepsilon/3 \quad \text{for } n \ge N.$$

Hence for $n \geq N$ and $x, y \in [a,b]$ with $|x-y| < \delta_0$,

$$|g_n(x) - g_n(y)| \leq |g_n(x) - g(x)| + |g(x) - g(y)| + |g(y) - g_n(y)|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Now the functions $g_1, \ldots, g_{N-1}$ are also uniformly continuous. Hence, there are $\delta_1, \ldots, \delta_{N-1}$ such that $|g_m(x) - g_m(y)| < \varepsilon$ for $x, y \in [a,b]$, $|x-y| < \delta_m$, for $m = 1, 2, \ldots, N-1$. Setting $\delta = \min\{\delta_0, \delta_1, \ldots, \delta_{N-1}\}$, we see that for $\varepsilon > 0$ there is a $\delta > 0$ such that <u>for all n</u>:

$$|g_n(x) - g_n(y)| < \varepsilon \quad \text{for } x, y \in [a,b] \text{ with } |x-y| < \delta.$$

The functions in K are uniformly continuous with the same ε and δ, sort of "uniformly uniformly continuous"!

This motivates

Book: Differential Operator example?

<u>Definition 7.4.2</u> Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subset X$, and F a set of functions from A into Y. F is <u>equicontinuous</u> <u>on</u> <u>A</u> if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $f \in F$,

$$d_Y(f(x), f(y)) < \varepsilon \text{ for all } x, y \in A \text{ with } d_X(x,y) < \delta.$$

—start—

<u>Example 7.4.6</u>

The functions in Example 7.4.5 are equicontinuous

<u>Example 7.4.7</u>

For fixed $L > 0$, let F be the set of functions

$$F = \left\{ f : [a,b] \to \mathbb{R} \;\middle|\; |f(x) - f(y)| \le L|x-y|, \text{ all } x, y \in [a,b] \right\},$$

that is, F is the set of Lipschitz continuous functions on $[a,b]$ with constant L. We also say that F is uniformly Lipschitz continuous with constant L. Then, F is equicontinuous (exercise).

<u>Example 7.4.8</u>

Consider

$$F = \{ x^n \mid n = 1, 2, 3, \ldots \}$$

on $[0,1]$. Take $\varepsilon = 1/2$.