Book
Th 8.3.2 is
stated for a nbhd
This is needed.

What we can show is

Theorem 8.3.2

If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $\{D_j f_i(x),\ 1 \le i \le m,\ 1 \le j \le n\}$ all exist in an open set G containing a point a and $D_j f_i(x)$ is continuous on G for $1 \le i \le m$, $1 \le j \le n$, then $Df(a)$ exists.

-start-

Proof

As for the proof of Theorem 8.3.1, it suffices to consider $m=1$ and $f: \mathbb{R}^n \to \mathbb{R}$. Then,

$$
\begin{aligned}
f(a+h) - f(a) = {} & f(a_1+h_1, a_2, \ldots, a_n) - f(a_1, \ldots, a_n) \\
& + f(a_1+h_1, a_2+h_2, a_3, \ldots, a_n) - f(a_1+h_1, a_2, a_3, \ldots) \\
& + \cdots \\
& + f(a_1+h_1, \ldots, a_n+h_n) \\
& \quad - f(a_1+h_1, \ldots, a_{n-1}+h_{n-1}, a_n).
\end{aligned}
$$

Now $D_1 f$ is the derivative of the function $f(x, a_2, \ldots, a_n)$. Applying the mean value theorem

$$f(a_1+h_1, a_2, \ldots, a_n) - f(a_1, \ldots, a_n) = h_1 \cdot D_1 f(\xi_1, a_2, \ldots, a_n)$$

for some ξ_1 between a_1 and a_1+h_1. The

i^{th} term above is

$$h_i D_i f(a_1+h_1, \ldots, a_{i-1}+h_{i-1}, \xi_i, a_{i+1}, \ldots, a_n) = h_i D_i f(c_i)$$

for some point c_i. Now

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - \sum_{i=1}^{n} D_i f(a) \cdot h_i \|}{\|h\|}$$

$$= \lim_{h \to 0} \frac{\| \sum_{i=1}^{n} (D_i f(c_i) - D_i f(a)) h_i \|}{\|h\|}$$

$$\leq \lim_{h \to 0} \sum_{i=1}^{n} \| D_i f(c_i) - D_i f(a) \| \frac{\|h_i\|}{\|h\|}$$

$$\leq \lim_{h \to 0} \sum_{i=1}^{n} \| D_i f(c_i) - D_i f(a) \|$$

$$= 0$$

by the continuity of $D_i f$, $1 \leq i \leq m$.

<u>Definition 8.3.3</u>

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $G \subset \mathbb{R}^n$ be an open set. If $\{D_j f_i(x), 1 \leq i \leq m, 1 \leq j \leq n\}$ exist on G and are continuous on G, we say that f is <u>continuously differentiable on G</u>.

Theorem 8.3.2 says that if f is continuously differentiable on an open set G, then f is differentiable at each point of G.

§8.4 A useful interpretation: $\mathcal{C}^1$

It is useful to interpret differentiation itself as a map. To do this, we recall some basic material on linear operators.

Definition 8.4.1

A map T from a vector space $\mathfrak{X}$ to a vector space $\mathfrak{Y}$ is *linear* if

(i) $T(x_1+x_2) = T(x_1) + T(x_2)$, $x_1, x_2 \in \mathfrak{X}$

(ii) $T(cx) = cT(x)$, $x \in \mathfrak{X}$, $c =$ scalar.

Note: this implies $T(0) = 0$.

Definition 8.4.2

We let $\mathcal{L}(\mathfrak{X},\mathfrak{Y})$ denote the vectorspace of all linear transformations from $\mathfrak{X}$ into $\mathfrak{Y}$.
If $\mathfrak{X}$ and $\mathfrak{Y}$ have norms $\| \, \|_{\mathfrak{X}}$ and $\| \, \|_{\mathfrak{Y}}$ respectively, then we can define a norm on $\mathcal{L}(\mathfrak{X},\mathfrak{Y})$ via

$$\|T\| = \sup_{\|x\|_{\mathfrak{X}}=1} \|Tx\|_{\mathfrak{Y}} = \sup_{x\neq 0} \frac{\|Tx\|_{\mathfrak{Y}}}{\|x\|_{\mathfrak{X}}},$$

$T \in \mathcal{L}(\mathfrak{X},\mathfrak{Y})$.

It is easy to see that $\| \ \|$ satisfies all the requirements for a norm and

$$\|Tx\|_{\mathbb{Y}} \le \|T\| \ \|x\|_{\mathbb{X}} \qquad \text{all } x \in \mathbb{X}.$$

We have the fundamental result

$\boxed{\text{Bode: put earlier as a metric space example}}$

Theorem 8.4.1

(a) If $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, then $\|T\| < \infty$ and T is uniformly continuous on $\mathbb{R}^n$.

(b) If $T, S \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar,

$$\|T+S\| \le \|T\| + \|S\|$$
$$\|cT\| \le |c| \ \|T\|.$$

If we define the metric $d(T,S) = \|T-S\|$, then $(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), d)$ is a metric space.

(c) If $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $S \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$, then $ST = S \circ T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ and

$$\|ST\| \le \|S\| \ \|T\|$$

(abuse of notation).

The proof is a collection of standard linear algebra results.

Bode: We need a layered review of linear algebra. First: structure of $\mathbb{R}^n$ in the beginning. At differentiation, structure of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

Recall that linear transformations between finite dimensional vector spaces can be represented as a matrix-vector product. If $\{x_1, \dots, x_n\}$ is a basis for a vector space $\mathbb{X}$ and $\{y_1, \dots, y_m\}$ is a basis for a vector space $\mathbb{Y}$, then $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ determines a unique set of $n \times m$ numbers via

$$T x_j = \sum_{i=1}^{m} a_{ij} y_i \quad , \quad 1 \le j \le n.$$

We write

$$[T] = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{m1} \cdots a_{mn} \end{pmatrix}$$

If $x \in \mathbb{X}$ has the expansion $x = \sum_{j=1}^{n} c_j x_j$,

then

$$T x = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} c_j \right) y_i ,$$

so the coordinates of Tx with respect to $\{y_1, \dots, y_m\}$ are $\sum_{j=1}^{n} a_{ij} c_j$, i.e, the coordinate vector of Tx is the product of $[T]$ with the coordinate vector of x.

Thus, there is a 1-1 correspondence between $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ and the set of $m \times n$ matrices.

Finally, the Schwartz inequality implies that if $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ then for $x \in \mathbb{R}^n$

$$\|Tx\|^2 \le \sum_{i,j} |a_{ij}|^2 \|x\|^2$$

or

$$\|T\| \le \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}.$$

This gives

<u>Theorem 8.4.2</u>

If S is a metricspace, $a_{11}, \ldots, a_{m,n}$ are real continuous functions on S, and for each $x \in S$, $T_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is the linear transformation with matrix

$$[T_x] = (a_{ij}(x))_{i=1,j=1}^{m,n}$$

then the map $x \to T_x$ is a continuous map of S into $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

In other words, if the matrix elements (a_{ij}) are continuous functions of some parameter then the corresponding linear map depends continuously on the parameter in the metric induced by the matrix norm.

We can now reword Theorem 8.3.2 — start

<u>Theorem 8.4.3</u> Let $f: \mathbb{R}^n \to \mathbb{R}^m$. f is continuously differentiable on an open set $G \subset \mathbb{R}^n$ if and only if Df is a continuous