Book: Lots' more examples

$$D(fg)(a) = g(a)\,Df(a) + f(a)\,Dg(a)$$

and if $g(a) \neq 0$,

$$D\left(\frac{f}{g}\right)(a) = \frac{g(a)\,Df(a) + f(a)\,Dg(a)}{(g(a))^2}.$$

We can use these results to compute derivatives.

<u>Example 8.2.1</u>

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be $f(x_1, x_2) = \sin(x_1 x_2^2)$. Here

Book: f is smooth, so we can go to Jacobian

$f = \sin \circ (\pi_1 \cdot (\pi_2)^2)$, so

$$f'(a_1, a_2) = \sin'(a_1 a_2^2) \cdot (a_2^2 \cdot (\pi_1)'(a_1, a_2) + a_1((\pi_2^2)')(a_1, a_2))$$

$$= \sin'(a_1 a_2^2) \cdot (a_2^2 (\pi_1)'(a_1, a_2) + 2a_1 a_2 (\pi_2)'(a_1, a_2))$$

$$= \cos(a_1 a_2^2) \cdot (a_2^2 (1, 0) + 2a_1 a_2 (0, 1))$$

$$= (a_2^2 \cos(a_1 a_2^2),\ 2a_1 a_2 \cos(a_1 a_2^2))$$

§8.3 Partial Derivatives

– start –

We start by considering "one variable at a time" when considering differentiation of a function from $\mathbb{R}^n$ to $\mathbb{R}$.

<u>Definition 8.3.1</u>

Let $f: \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. The limit

$$\lim_{h \to 0} \frac{f(a_1, a_2, \ldots, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, a_2, \ldots, a_i, \ldots, a_n)}{h}$$

if it exists, is denoted

$$D_i f(a) = \frac{\partial f}{\partial x_i}(a)$$

and is called the <u>i^{th} partial derivative of f at a.</u>

Note that $D_i f(a)$ is the usual calculus derivative of the function

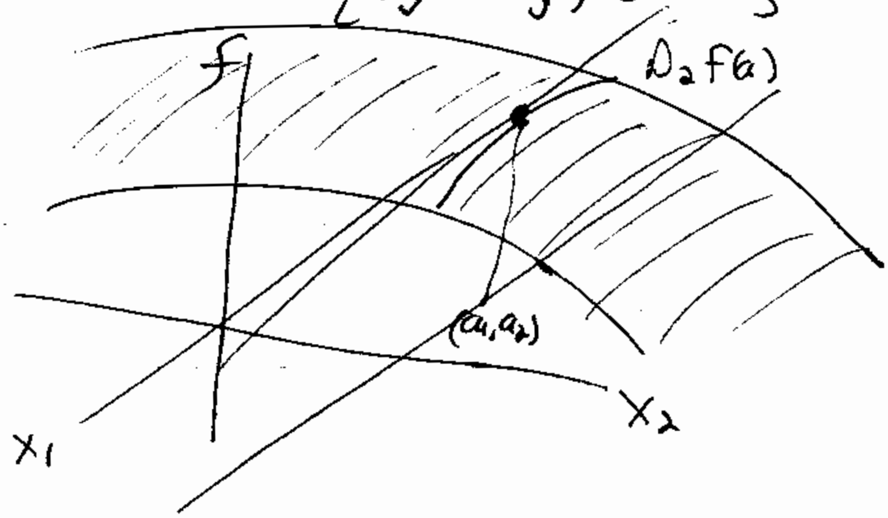
$$g(x) = f(a_1, .., a_{i-1}, x, a_i, .., a_n),$$

i.e.,

$$D_i f(a) = \frac{d g(a_i)}{dx} = g'(a_i).$$

This means that $D_i f(a)$ is the slope of the tangent line at $(a, f(a))$ to the curve obtained by intersecting the graph of f with the plane

$$\{x_j = a_j,\ j \neq i\}.$$



So we know how to compute $D_i f(a)$ in this case using the standard calculus defn. If $f(x_1,..,x_n)$ is given by a formula involving $x_1,..,x_n$, we find $D_i f(x_1,..,x_n)$ by differentiating the function of a scalar obtained by treating x_j as constant for $j \neq i$ and x_i as the variable.

Example 8.3.1

$$f(x_1, x_2) = \sin(x_1 x_2^2)$$
$$D_1 f(x_1, x_2) = \cos(x_1 x_2^2)\, x_2^2$$
$$D_2 f(x_1 x_2) = \cos(x_1 x_2^2)\, 2x_1 x_2$$

Example 8.3.2

$$f(x_1, x_2) = x_1^{x_2}$$
$$D_1 f(x_1, x_2) = x_2 \cdot x_1^{x_2 - 1}$$
$$D_2 f(x_1, x_2) = x_1^{x_2} \cdot \log(x_1).$$

Definition 8.3.2 If $f: \mathbb{R}^n$ and $\mathbb{R}$ has the property that $D_i f(x)$ exists for all $x \in \mathbb{R}^n$ we obtain a function $D_i f: \mathbb{R}^n \to \mathbb{R}$ called the

<u>partial derivative (function) of f.</u>

We now consider the case $f : \mathbb{R}^n \to \mathbb{R}^m$. We can consider each component function $f_i : \mathbb{R}^n \to \mathbb{R}$, where $f = (f_1, \ldots, f_m)$.

<u>Theorem 8.3.1</u> If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then $D_j f_i(a)$ exists for $1 \le i \le m$, $1 \le j \le n$ and

$$f'(a) = \left(D_j f_i(a) \right)_{i=1, j=1}^{m, n} = \begin{pmatrix} D_1 f_1 & D_2 f_1 \cdots & D_n f_1 \\ D_1 f_2 & \vdots & \vdots \\ \vdots & & \\ D_1 f_m & D_2 f_m & D_n f_m \end{pmatrix}$$

<u>Proof</u>

Suppose $m = 1$, i.e., $f : \mathbb{R}^n \to \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}^n$ be defined

$$g(x) = (a_1, a_2, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n).$$

Then,

$$D_j f(a) = (f \circ g)'(a_j)$$

$$= f'(a) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th} \text{ place.}$$

Now $(f \circ g)'(a_j)$ has a single entry $D_j f(a)$

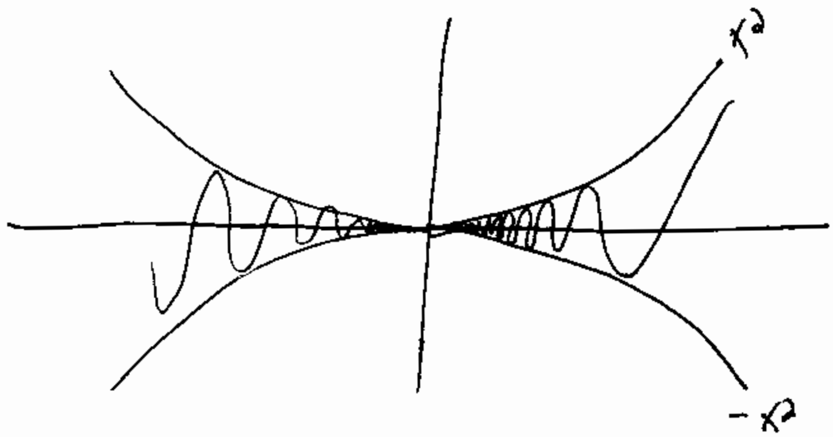
which shows that $D_j f(a)$ exists and is the j^{th} entry of the $1 \times n$ matrix $f'(a)$.
For $f: \mathbb{R}^n \to \mathbb{R}^m$, we use Theorem 8.2.4 to conclude that each f_i is differentiable and the i^{th} row of $f'(a)$ is $(f_i)'(a)$.

Unfortunately, the converse is false. The root of the trouble is that the derivative of a differentiable function need not be continuous.

Example 8.3.3

Consider the continuous function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$



For $x \neq 0$, $f'(x) = -\cos(\frac{1}{x}) + 2x\sin(\frac{1}{x})$. While defined for $x \neq 0$, $\lim_{x \to 0} f'(x)$ is undefined.

However,

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x}) - 0}{x - 0} = 0$$

is defined and $f'(0) = 0$. So $f'(x)$ is not continuous at 0, though it is defined everywhere.

We can construct examples that show the existence of $\{D_j f_i(a)\}_{i=1,\,j=1}^{m,\,n}$ is not sufficient to guarantee that f is differentiable at a.

<u>Example 8.3.4</u>

The function

$$f(x,y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases}$$

is continuous and has first order partial derivatives on $\mathbb{R}^2$, but is not differentiable at $(0,0)$. (Homework Problem).

Book Th 8.3.2 is Stated for a nbhd. This is needed.

What we can show is

Theorem 8.3.2

If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $\{D_j f_i(x),\ 1 \le i \le m,\ 1 \le j \le n\}$ all exist in an open set G containing a point a and $D_j f_i(x)$ is continuous on G for $1 \le i \le m$, $1 \le j \le n$, then $Df(a)$ exists.

— start —

Proof

As for the proof of Theorem 8.3.1, it suffices to consider $m=1$ and $f: \mathbb{R}^n \to \mathbb{R}$. Then,

$$
\begin{aligned}
f(a+h) - f(a) = {} & f(a_1+h_1, a_2, \dots, a_n) - f(a_1, \dots, a_n) \\
& + f(a_1+h_1, a_2+h_2, a_3, \dots, a_n) - f(a_1+h_1, a_2, a_3, \dots) \\
& + \cdots \\
& + f(a_1+h_1, \dots, a_n+h_n) \\
& \quad - f(a_1+h_1, \dots, a_{n-1}+h_{n-1}, a_n).
\end{aligned}
$$

Now $D_1 f$ is the derivative of the function $f(x, a_2 \dots, a_n)$. Applying the mean value theorem

$$f(a_1+h_1, a_2 \dots, a_n) - f(a_1, \dots, a_n) = h_1 \cdot D_1 f(\xi_1, a_2 \dots, a_n)$$

for some ξ_1 between a_1 and $a_1 + h_1$. The

i^{th} term above is

$h_i D_i f(a_1+h_1, \dots, a_{i-1}+h_{i-1}, \xi_i, a_{i+1}, \dots, a_n) = h_i D_i f(c_i)$