Example 4.3.2

Consider $\mathbb{Q} \subset (\mathbb{R}, |\,|)$. $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a Cauchy sequence in $\mathbb{Q}$ because we know that $\left(1+\frac{1}{n}\right)^n \to e^1$ in $(\mathbb{R}, |\,|)$, but its limit $e \notin \mathbb{Q}$.

Example 4.3.3

Consider the space of polynomials on $[a,b]$: $\mathcal{P}([a,b]) \subset C([a,b])$. The sequence $\left\{1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right\}$ converges uniformly to e^x on $[a,b]$, i.e.

$$\max_{a \le x \le b} \left|1+x+\cdots+\frac{x^n}{n!} - e^x\right| \xrightarrow[n\to\infty]{} 0$$

Exp 6?

by Taylor's theorem. Hence, $\left\{1+x+\cdots+\frac{x^n}{n!}\right\}$ is a Cauchy sequence in $\mathcal{P}([a,b])$, but its limit $e^x \notin \mathcal{P}([a,b])$.

Back: sum up the examples in words the Cauchy sequence acts like it converges but its limit is not in the space and the defin cannot be verified

— Start —

Definition 4.3.1

A metric space is complete if every Cauchy sequence converges to an element in the space.

Compactness is a property that has to be established, and this may or may not be easy to do!

As a first example, we prove

Theorem 4.3.1

$\mathbb{R}^n$ is complete.

Proof

Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^n$. It is easy to prove that any Cauchy sequence in a metric space is bounded. Hence, $\{x_n\}$ is contained in a compact n-cell. This means that it has a convergent subsequence and by Theorem 4.2.2 converges itself.

book make theorem earlier?

As a second example, we present

Example 4.3.4

Recall that a real valued function f is bounded on an interval $[a,b]$ if there is a constant M such that

$$|f(x)| \le M \quad \text{for} \quad a \le x \le b.$$

Also, a continuous function is bounded on a finite closed interval, but a function that is bounded does not have to be continuous.

book make a definition?

We define

$$\mathcal{M}([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is bounded} \}$$

Book do this as example of a metric space

and introduce the metric

$$d(f,g) = \sup_{[a,b]} |f(x) - g(x)| \quad , f, g \in \mathcal{M}([a,b])$$

It is easy to show $(\mathcal{M}([a,b]), d)$ is a metric space (Problem).

We show that $\mathcal{M}([a,b])$ is complete.

book holds in general

This is a three step process:

1) Find a natural candidate for a limit for a Cauchy sequence

2) Verify the limit is the metric space

3) Prove the Cauchy sequence converges

to this limit.

Let $\{f_k\}$ be a Cauchy sequence in $\mathcal{M}([a,b])$. For a fixed x in $[a,b]$, consider the sequence of numbers $\{f_k(x)\}$. This is a Cauchy sequence in $(\mathbb{R}, |\,|)$, since

$$|f_n(x) - f_m(x)| \le \sup_{a \le y \le b} |f_n(y) - f_m(y)| = d(f_n, f_m).$$

Since $\mathbb{R}$ is complete, $\{f_n(x)\}$ converges to a real number. Define the function $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad a \le x \le b.$$

This is our candidate for the limit. We next show $f \in \mathcal{M}([a,b])$, i.e., f is bounded. Choose N so that $d(f_m, f_n) \le 1$ for $n, m \ge N$. In particular,

$$\sup_{a \le x \le b} |f_N(x) - f_m(x)| \le 1 \quad \text{for } m \ge N.$$

Book: can you rig up a space with picture complete that is not complete?

Book: discuss completeness vs closure

We let $m \to \infty$ in this inequality and obtain

$$\sup_{a \le x \le b} |f_N(x) - f(x)| \le 1.$$

Since $f_N \in \mathcal{M}([a,b])$, there is an M such that $\sup_{a \le x \le b} |f_N(x)| \le M$ and so

$$\sup_{a \le x \le b} |f(x)| \le M + 1.$$

Book: emphasize $f \in \mathcal{M}$ we need to be able to write down $d(f, f_n)$ otherwise might not be defined.

Finally, we show $\{f_n\} \to f$ in $(\mathcal{M}([a,b]), d)$. Note, that we have pointwise convergence by construction, but we do not have uniform convergence, which is the convergence notion in $(\mathcal{M}([a,b]), d)$, automatically.

Let $\varepsilon > 0$. There is an N such that

$$d(f_m, f_N) < \varepsilon \quad \text{for } m \ge N.$$

This means

$$|f_m(x) - f_N(x)| < \varepsilon$$

for $a \le x \le b$ and $m \ge N$. Taking the limit as $m \to \infty$ yields

$$|f(x) - f_N(x)| < \varepsilon \quad , \quad a \le x \le b.$$

For $n \ge N$,

$$|f_n(x) - f(x)| \le |f_n(x) - f_N(x)| + |f_N(x) - f(x)|$$
$$< 2\varepsilon,$$

for $a \le x \le b$. Hence, $d(f_n, f) < 2\varepsilon$ for $n \ge N$.

We can characterize completeness in terms of a generalization of the Cantor Intersection Property, recall Defn. 3.1.7.

Back Pictures & examples

<u>Definition 4.3.2</u>

Let A be a nonempty subset of a metric space $(\mathfrak{X}, d)$. The <u>diameter</u> of A is defined by

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y).$$

Note that $\operatorname{diam}(A)$ can be infinite and the diameter of a set consisting of a single point is zero.

We have

Theorem 4.3.2

A metric space is complete if and only if the intersection of every descending sequence of nonempty, closed sets whose diameters approach zero consists of a single point.

ck? if diamters → 0 - then what?

Book application

is Cauchy, $x_n \to x$.

We can characterize complete subsets of a complete metric space very nicely

Theorem 4.3.3

Let $(\mathbb{X}, d)$ be a complete metric space. A subspace $\mathbb{Y} \subset \mathbb{X}$ is complete if and only if $\mathbb{Y}$ is closed.

Proof

Suppose $\mathbb{Y}$ is closed and $\{x_n\}$ is a Cauchy sequence in $\mathbb{Y}$. Since $\mathbb{X}$ is complete, $x_n \to x \in \mathbb{X}$. But, $\mathbb{Y}$ is closed, so $x \in \mathbb{Y}$. This means $\mathbb{Y}$ is complete.

If $\mathbb{Y}$ is complete, let x be a limit point of $\mathbb{Y}$. There is a sequence $\{x_n\}$ in $\mathbb{Y}$ with $x_n \to x$. This sequence converges in $\mathbb{X}$, so it is a Cauchy sequence in $\mathbb{X}$, and therefore in $\mathbb{Y}$. This means $x_n \to \tilde{x} \in \mathbb{Y}$. Since limits are unique $x = \tilde{x} \in \mathbb{Y}$ and $\mathbb{Y}$ is closed.