of $\{x_n\}$.

Choose n_1 so that $x_{n_1} \neq x$. If no such n_1 exists, then A has one point, and we are done. Let $\delta = d(x, x_1)$. Suppose $n_1, \ldots, n_{m-1}$ are chosen. Since x is a limit point of A, there is a $y \in A$ with $d(x, y) < 2^{-m}\delta$. Since $y \in A$, there is an $n_m > n_{m-1}$ with $d(y, x_{n_m}) < 2^{-m}\delta$. We have

$$
\begin{aligned}
d(x, x_{n_m}) &\leq d(x, y) + d(y, x_{n_m}) \\
&\leq 2^{-m}\delta + 2^{-m}\delta = 2^{1-m}\delta.
\end{aligned}
$$

We conclude that $\{x_{n_m}\} \to x$.

Finally, we discuss the connection between boundedness (Defn. 2.3.9) and convergence of sequences. (This material belongs after Thm 2.3.6)

<u>Definition 4.1.1</u>

A sequence $\{x_n\}$ in a metric space (X, d) is <u>bounded</u> if its range forms a bounded set in X. Otherwise, it is <u>unbounded</u>.

Equivalently, $x_n \in A$, A a bounded subset in $\mathbb{X}$, for all n.

<u>Example 4.1.1</u> In $(\mathbb{R}, |\,|)$,

(1) $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges and is bounded

(2) $\{n^2\}_{n=1}^{\infty}$ diverges and is unbounded

(3) $\{1+(-1)^n\}_{n=1}^{\infty}$ diverges and is bounded

We have the relation

<u>Theorem 4.1.4</u>

A sequence in a metric space that converges is bounded.

<u>Proof</u> Suppose $\{x_n\}$ is a sequence in a metric space $(\mathbb{X}, d)$ that converges to x. There is an integer N such that $d(x_n, x) < 1$ for $n \geq N$. Set

$$r = \max\{1, d(x, x_1), \ldots, d(x, x_{N-1})\}.$$

Then, $d(x_n, x) \leq r$ for all n.

---

§4.2 Cauchy Sequences

The practical trouble with the standard definition of convergence is that it involves the (usually) unknown limit.

Example 4.2.1

The sequence

$$\left\{ \sum_{m=1}^{n} \frac{4}{n} \sqrt{e^{m\frac{4}{n}} - \sin\left(\frac{2}{1+m\frac{4}{n}}\right)} \right\}_{n=1}^{\infty}$$

converges, because it converges to

$$\int_1^4 \sqrt{e^x - \sin\left(\frac{2}{1+x}\right)}\, dx,$$

which we can prove exists by standard Calculus results. However, we do not know the value of this integral and cannot verify the definition of convergence.

The notion of a Cauchy sequence gives a way around this difficulty.

The idea is that if a sequence converges to a limit, i.e., the terms in the sequence approach a limit as the index increases, then the terms also approach each other as the index increases.

Definition 4.2.1

A sequence $\{x_n\}$ in a metric space (X, d) is a Cauchy sequence if for every $\varepsilon > 0$ there is an $N > 0$ such that $d(x_n, x_m) < \varepsilon$ for $n, m \geq N$.

Example 4.2.2

$\{\frac{1}{n}\}$ is a Cauchy sequence in $C(\mathbb{R}, |\,|)$.

For,

$$d(\tfrac{1}{n}, \tfrac{1}{m}) = \left|\tfrac{1}{n} - \tfrac{1}{m}\right| \leq \frac{2}{\min\{n,m\}}$$

so given $\varepsilon > 0$, if we choose $N > \frac{2}{\varepsilon}$, then for $n, m \geq N$, $|\frac{1}{n} - \frac{1}{m}| < \varepsilon$.

Book: better example

Example 4.2.3

$\{\frac{\sin(nx)}{n}\}$ is a Cauchy sequence in $C([0,\pi])$

since

$$\left|\frac{\sin(nx)}{n} - \frac{\sin(mx)}{m}\right| \leq \frac{2}{\min\{n,m\}}, \quad 0 \leq x \leq \pi,$$

hence given $\varepsilon > 0$, if $N > \frac{1}{\varepsilon}$, then

$$d\left(\frac{\sin(nx)}{n}, \frac{\sin(mx)}{m}\right) < \varepsilon \quad \text{for } n, m \geq N.$$

Fitting our intuition:

Theorem 4.2.1

Any sequence in a metric space that converges is a Cauchy sequence.

Proof

Assume $\{x_n\}$ is a sequence in a metric space $(\mathfrak{X}, d)$ and $x_n \to x$. Choose $\varepsilon > 0$. There is an N such that $d(x_n, x) < \varepsilon$ for $n \geq N$. Hence,

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \leq 2\varepsilon, \quad n, m \geq N,$$

and $\{x_n\}$ is Cauchy.

back in chap

Moreover

Theorem 4.2.2

If a subsequence of a Cauchy sequence in a metric space converges to a limit, then the Cauchy sequence itself converges to the same limit.

Proof

Let $\{x_n\}$ be a Cauchy sequence

in a metric space (X, d). Suppose the subsequence $\{x_{n_k}\}$ converges to x. Given $\varepsilon > 0$, choose N so that $d(x_n, x_m) < \varepsilon/2$ for $n, m \geq N$. Choose K such that $k \geq K$ implies $n_k \geq N$ and $d(x_{n_k}, x) < \varepsilon/2$. Then for all $n \geq N$ and $k \geq K$,

$$d(x_n, x) < d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon.$$

Back: Cauchy seq is bounded

§4.3 Completeness

Unfortunately, the converse to Theorem 4.2.1 just does not hold. Not every Cauchy sequence in a metric space must converge to a point in the space.

Example 4.3.1

Consider $(0,1) \subset (\mathbb{R}, |\,|)$. $\{\frac{1}{n}\}$ is a Cauchy sequence in $(0,1)$, but does not converge to a limit in $(0,1)$.

Example 4.3.2

Consider $\mathbb{Q} \subset (\mathbb{R}, |\,|)$. $\{(1+\frac{1}{n})^n\}$ is a Cauchy sequence in $\mathbb{Q}$ because we know that $(1+\frac{1}{n})^n \to e^1$ in $(\mathbb{R}, |\,|)$, but its limit $e \notin \mathbb{Q}$.

Example 4.3.3

Consider the space of polynomials on $[a,b]$: $\mathcal{P}([a,b]) \subset C([a,b])$. The sequence $\{1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\}$ converges uniformly to e^x on $[a,b]$, i.e.

$$\max_{a \le x \le b} \left|1+x+\cdots+\frac{x^n}{n!}-e^x\right| \xrightarrow[n\to\infty]{} 0$$

by Taylor's theorem. Hence, $\{1+x+\cdots+\frac{x^n}{n!}\}$ is a Cauchy sequence in $\mathcal{P}([a,b])$, but its limit $e^x \notin \mathcal{P}([a,b])$.

Book: sum up the examples in words: the Cauchy sequence acts like it converges but its limit is not in the space and the definition cannot be verified

Definition 4.3.1

A metric space is <u>complete</u> if every Cauchy sequence converges to an element in the space.