Definition 7.2.1

Let (X, d_X), (Y, d_Y) be metric spaces, $A \subseteq X$, and $\{f_n\}$ a sequence of functions $f_n : A \to Y$ for all n. $\{f_n\}$ <u>converges uniformly to f on A</u> if for every $\varepsilon > 0$ there is an N such that

$$d_Y(f_n(x), f(x)) < \varepsilon \quad \text{for } x \in A \text{ and } n \geq N.$$

Please compare to Def. 7.1.1.

—Start—

Example 7.2.2

The functions in Ex. 7.1.1 converge uniformly to 0 since $\left|\frac{1}{n}\sin(nx) - 0\right| \leq \frac{1}{n}$ for all $0 \leq x \leq \pi$.

Example 7.2.3

The functions in Ex. 7.1.2 do not converge uniformly on $[0,1]$ since

$$\sup_{0 \leq x \leq 1} |f_n(x) - 0| = n$$

Example 7.2.4

The sequence $\{x^n\}$ does not converge uniformly to $\chi_1(x)$ on $[0,1]$, but does on $[0,\frac{1}{2}]$.

Uniform convergence goes well with continuity.

Theorem 7.2.1

Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subset X$. Suppose $\{f_n\}$ is a sequence of functions with $f_n : A \to Y$ continuous on A for all n and $f_n \to f : A \to Y$ uniformly on A. Then, f is continuous on A.

Proof

Choose $x \in A$ and $\varepsilon > 0$. We want to show we can make $d_Y(f(y), f(x))$ smaller than ε by making $d_X(x,y)$ small. Uniform convergence means that we can make $d_Y(f(x), f_n(x))$ and

$d_Y(f(y), f_n(y))$ small, so for $x, y \in A$ we write

$$d_Y(f(x), f(y)) \le d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(y), f(y)).$$

By uniform convergence, there is an N such that $d_Y(f(x), f_n(x)) < \varepsilon$ and $d_Y(f(y), f_n(y)) < \varepsilon$ for $n \ge N$, independent of x and y. Since f_n is continuous on A, there is a $\delta > 0$ such that for any <u>fixed</u> $n \ge N$,

$$d(f_n(x), f_n(y)) < \varepsilon,$$

for all $y \in A$, $d_X(x, y) < \delta$. Hence, using that value of n, we conclude

$$d(f(x), f(y)) < 3\varepsilon \quad \text{for all } y \in A,\ d_X(x,y) < \delta.$$

The functions in Ex. 7.1.2 show the converse does not hold: a sequence of continuous functions can converge to a continuous function without the convergence being uniform.

We now discuss the related topic of completeness. We state the Cauchy criterion for uniform convergence.

Theorem 7.2.2

Let (X, d_X) and (Y, d_Y) be metric spaces, Y complete, $A \subset X$, and $\{f_n\}$ a sequence with $f_n : A \to Y$ for all n. $\{f_n\}$ converges uniformly on A if and only if for every $\varepsilon > 0$ there is an N such that

$$d_Y(f_n(x), f_m(x)) < \varepsilon \quad \text{for } n, m \geq N \text{ and } x \in A.$$

Proof

Suppose $\{f_n\}$ converges uniformly on A

to f. Given $\varepsilon > 0$, there is an N such that

$$d_Y(f_n(x), f(x)) < \varepsilon \quad , \quad x \in A, n \geq N.$$

So

$$d_Y(f_n(x), f_m(x)) \leq d_Y(f_n(x), f(x)) + d_Y(f(x), f_m(x)) < 2\varepsilon$$

for $x \in A$, $n, m \geq N$.

Conversely, suppose the Cauchy condition holds. The sequence of points $\{f_n(x)\}$ is a Cauchy sequence in Y for each $x \in A$, and therefore has a limit in Y that we call $f(x)$. This defines $f: A \to Y$. $\{f_n\}$ converges pointwise to f on A and we have to show the convergence is uniform. Let $\varepsilon > 0$ and choose N so

$$d_Y(f_n(x), f_m(x)) < \varepsilon/2 \quad \text{for } n, m \geq N, x \in A.$$

Fix n and let $m \to \infty$. Since $f_m \to f$ as $m \to \infty$ and $d_Y(f(x), \cdot)$ is continuous,

$d_Y(f_n(x), f(x)) \le \varepsilon/2 < \varepsilon$ for $n \ge N$, $x \in A$.

Now, we observe that convergence in $C([a,b])$ is uniform convergence and $\mathbb{R}$ is complete. Theorems 7.2.1 and 7.2.2 imply

Theorem 7.2.3

$C([a,b])$ is closed and complete.

§7.3 $C([a,b])$ is Separable

We know $\mathbb{R}$ is separable and, in particular, $\mathbb{Q}$ is dense in $\mathbb{R}$. This is extremely important from a practical point of view because it means we can approximate irrational numbers using rational numbers. This is what makes large scale scientific computing possible, for example.

We prove that $C([a,b])$ is separable by first showing that continuous functions can be approximated arbitrarily well by polynomials. $\boxed{\text{Back: Present 2 proofs, 1 nonconstructive?}}$

Theorem 7.3.1 **Weierstrass Approximation Theorem**

Assume that f is continuous on $[a,b]$. Given $\varepsilon > 0$, there is a polynomial P_n of sufficiently high degree n such that

$$d(f,P_n) = \sup_{a \le x \le b} |f(x) - P_n(x)| < \varepsilon.$$

Another way to state this result is that there is a sequence of polynomials $\{P_n\}$ (of course in $C([a,b])$) that converges to f in $C([a,b])$, that is, uniformly.

This theorem is profoundly important.

It is the reason, for example, that the use of polynomials is so widespread in numerical analysis, i.e., approximation of functions, integrals, solutions of differential equations, and so on. Note, unlike Taylor's polynomials, this result does not require increasing smoothness of f to increase the accuracy of the polynomial approximations.

— start —

To prove that $C([a,b])$ is separable, we first note that if

$$P(x) = \sum_{m=0}^{n} a_m x^m$$

and

$$\tilde{p}(x) = \sum_{m=0}^{n} \hat{a}_m x^m$$

are two polynomials on $[a,b]$, then

$$d(P,\tilde{P}) = \sup_{a \le x \le b} |P(x) - \tilde{P}(x)|$$

$$\le C \cdot \sum_{m=0}^{n} |a_m - \hat{a}_m|$$

$$\le (n+1) \cdot C \cdot \max_{0 \le m \le n} |a_m - \hat{a}_m|$$

where C is a constant that depends on