$x_{n_k} \in G_M$ for all sufficiently large k. But, this contradicts the construction by which $x_{n_k} \notin G_M$ for $n_k > M$.

—start—

§3.5 Some properties of compactness

We now show some easy, but characteristic properties of compactness of subsets. The first is a homework problem

Theorem 3.5.1

Let $(\mathbb{X}, d)$ be a metric space.

(1) If $\{K_1, .., K_n\}$ are compact subsets of $\mathbb{X}$, then $\bigcup_{m=1}^{n} K_m$ is compact.

(2) If $\{K_\alpha\}_{\alpha \in a}$ is a collection of compact subsets of $\mathbb{X}$, then $\bigcap_{\alpha \in a} K_\alpha$ is compact.

Recall the "flaw" concerning openness and subsets discussed in Example 2.3.12. If $(\mathbb{X}, d)$ is a metric space and $\mathbb{Y} \subset \mathbb{X}$, then $(\mathbb{Y}, d)$ is a metric space. A set $G \subset \mathbb{Y}$ may be open in $\mathbb{Y}$, but this does not mean

, it is open in $\mathbb{X}$. Contrast this to

Theorem 3.5.2

Suppose $(\mathbb{X}, d)$ is a metric space and $K \subset \mathbb{Y} \subset \mathbb{X}$. K is a compact subset of $\mathbb{X}$ if and only if K is a compact subset of $\mathbb{Y}$.

Proof

Suppose $K \subset \mathbb{X}$ is compact and let $\{A_\alpha\}_{\alpha \in a}$ be a collection of sets that are open relative to $\mathbb{Y}$ such that $K \subset \bigcup_{\alpha \in a} A_\alpha$. By Theorem 2.3.7, there are open sets $\{G_\alpha\}_{\alpha \in a}$ in $\mathbb{X}$ with

Book: explain what we are doing!

$$A_\alpha = \mathbb{Y} \cap G_\alpha \quad , \alpha \in a.$$

Since K is compact in $\mathbb{X}$ and is covered by $\{A_\alpha\}_{\alpha \in a}$,

$$K \subset G_{\alpha_1} \cup \ldots G_{\alpha_n} \qquad \text{some } \alpha_1, \ldots, \alpha_n \in a.$$

But, this implies

$$K \subset A_{\alpha_1} \cup \ldots \cup A_{\alpha_n},$$

and K is compact in $\mathbb{Y}$.

If $K \subset Y$ is compact in Y, let $\{G_\alpha\}_{\alpha\in a}$ be a collection of open sets in X that covers K. Set $A_\alpha = Y \cap G_\alpha$, $\alpha \in a$. $\{A_\alpha\}_{\alpha\in a}$ are open relative to Y, and the collection covers K, so by compactness

$$K \subset A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}, \quad \text{some } \alpha_1,.., \alpha_n \in a.$$

Hence,

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$$

and K is compact in X.

Finally, recall that Example 3.2.3 shows that being closed and bounded is not sufficient to guarantee compactness. Interestingly,

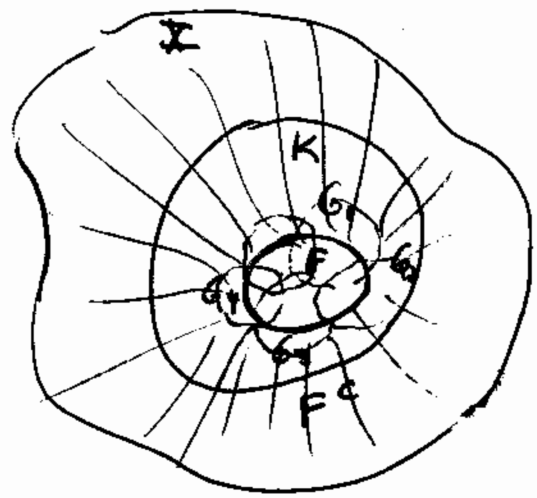
<u>Theorem 3.5.3</u>

Closed subsets of a compact set in a metric space are compact.

<u>Proof</u> Let (X,d) be a metric space, $K \subset X$ compact, and $F \subset K$ closed.

Let $\{G_\alpha\}_{\alpha \in a}$ be an open cover of F.

If F^c is added to $\{G_\alpha\}_{\alpha \in a}$, we obtain an open cover $\{\{G_\alpha\}_{\alpha \in a} \cup F^c\}$ of K.



Since K is compact, there is a finite subcollection from $\{G_\alpha\}_{\alpha \in a} \cup \{F^c\}$ that covers K, and hence F. We can remove F^c and still retain a cover of F. Thus, a finite subcollection of $\{G_\alpha\}_{\alpha \in a}$ covers F, and F is compact.

§3.6 Compact Sets in $\mathbb{R}^n$

As a special case, we consider $\mathbb{R}^n$. First, we prove the generalization of the closed bounded interval is compact.

Definition 3.5.1

Let $a_m < b_m$, $m = 1, 2, .., n$, be numbers in $\mathbb{R}$. The set

$$\{X = (x_1, .., x_n) \in \mathbb{R}^n \mid a_m \leq x_m \leq b_m \ , \ 1 \leq m \leq n\}$$

is an <u>n-cell</u> in $\mathbb{R}^n$.

(Book: pictures)

Theorem 3.5.1

Every n-cell of $\mathbb{R}^n$ is compact. This implies in particular that $[a,b] \subset \mathbb{R}$, $a<b$, is compact.

Proof

We first show a modified form of the finite intersection property for n-cells. We begin with intervals in $\mathbb{R}^1$:

Let $\{I_m\}_{m=1}^{\infty}$ be a descending sequence of closed intervals in $\mathbb{R}^1$, i.e., $I_1 \supset I_2 \supset \ldots$. This implies $\{I_m\}_{m=1}^{\infty}$ has the finite intersection property. We prove that $\bigcap_{m=1}^{\infty} I_m$ is nonempty.

Let $I_m = [a_m, b_m]$. The sequence $\{a_m\}$ is bounded above by b_1, hence

$$X = \sup_m a_m < \infty.$$

We show that $X \in \bigcap_{m=1}^{\infty} I_m$. For positive integers m, ℓ;

$$a_\ell \le a_{\ell+m} \le b_{\ell+m} \le b_m ;$$

so $X \le b_m$ for all m. Since $a_m \le X$ for all m, $a_m \le X \le b_m$ for all m, and $X \in \bigcap_m I_m$.

Book only $\mathbb{R}$ specific result

—start—

Now suppose $\{I_m\}_m$ is a descending sequence of n-cells in $\mathbb{R}^n$. Let

$$I_m = \{x \mid a_{m,\ell} \le x_\ell \le b_{m,\ell},\ 1 \le \ell \le n, m = 1, 2, \ldots\}$$

Set

$$I_{m,\ell} = [a_{m,\ell}, b_{m,\ell}] \subset \mathbb{R}^1$$

$1 \le \ell \le n$, $m = 1, 2, \ldots$. In other words

$$I_m = I_{m,1} \times \cdots \times I_{m,n}$$

For each ℓ, $\{I_{m,\ell}\}_m$ is a descending sequence of intervals in $\mathbb{R}^1$. Hence, there are real numbers X_ℓ such that

$$a_{m,\ell} \le X_\ell \le b_{m,\ell}, \quad 1 \le \ell \le n, m = 1, 2, \ldots$$