M517: Hints for Problem Set 5

1. Use the definition of continuity involving a sequence.
2. Prove there is a $\delta > 0$ such that $f(x) \ge \delta$ for $a \le x \le b$. If this is not true, then for every m there is an x_m such that $0 < f(x_m) < 1/m$. Get a contradiction.
3. What are the maximum and minimum values of these functions on $\mathbb{R}$?
4. $f_n(x_n) - f(x) = \left(f_n(x_n) - f(x_n)\right) + \left(f(x_n) - f(x)\right)$
6. Explore the Lipschitz continuity properties.
7. For (a): to do this problem, show that the set $\mathbf{L}$ of **all** functions that are Lipschitz continuous on $[a,b]$ with constant L, i.e. $\mathbf{L} = \{ f : [a,b] \to \mathbb{R} : | f(x) - f(y) | \le L | x - y |, \text{ all } x, y \in [a,b]\}$ is equicontinuous. Conclude the same for $\{f_n\}$ because it is a subset of $\mathbf{L}$. For (b): You should use the Arzela-Ascoli theorem 7.4.2. To do this, you must first find a closed subset of $C([a,b])$ that is uniformly bounded and equicontinuous that contains the sequence $\{f_n\}$. Now given $x \in [a,b]$, define the space $\mathbf{L}$ to be
   $$\mathbf{L} = \{ f : [a,b] \to \mathbb{R} : | f(z) - f(y) | \le L | z - y |, \text{ all z}, y \in [a,b] \text{ and } | f(x) | \le M \}$$
   for some constant M. Show that $\mathbf{L}$ has the desired properties. To show that $\mathbf{L}$ is closed, show that it contains its limit points. This means: show that if f is a limit point of $\mathbf{L}$ then f is in $\mathbf{L}$, i.e. is bounded by M at x and is Lipschitz continuous with constant L. To prove the second property, for example, if f is a limit point of $\mathbf{L}$ there is a sequence $\{g_n\}$ that converges uniformly to f. Now $f(x) - f(y) = f(x) - g_n(x) + g_n(x) - g_n(y) + g_n(y) - f(y)$ which implies an estimate of the form $| f(x) - f(y) | \le L | x - y | + 2\varepsilon$ for $\varepsilon > 0$ arbitrarily small. You also need to use (a) (which is why I suggested doing (a) a little differently than stated in the problem. For (c): Use (a) and (b).
8. First use the expansion $f(x) - f(y) = f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)$ to show that f is continuous. For $\varepsilon > 0$ there is a $\delta > 0$ such that $\left| f_n(x) - f_n(y) \right| < \varepsilon$ and $\left| f(x) - f(y) \right| < \varepsilon$ for $| x - y | < \delta$ and all n. Choose a mesh $\{x_1, \cdots, x_n\}$ with spacing between nodes smaller than δ. Demonstrate uniform convergence on this finite set of points. For a general f and x, choose x_i closer than δ to x. Use the expansion
   $$f_n(x) - f(x) = f_n(x) - f_n(x_i) + f_n(x_i) - f(x_i) + f(x_i) - f(x)$$