**Math 517, Assignment 5**
**Due Friday, November 21**
*Remember to provide full reasoning for all answers!*

1. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous and define $A = \{x : f(x) = 0\}$. Show $A$ is closed.

2. Show that if $f \in C([a, b])$ has the property that $f(x) > 0$ for $a \leq x \leq b$, then $1/f(x)$ is bounded on $[a, b]$.

3. Prove the sequence $\left\{ \dfrac{x}{n + x^2} \right\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$ (with the usual metric).

4. Let $(X, d_x)$ and $(Y, d_y)$ be metric spaces, $A \subset X$, and $\{f_n\}$ a sequence of continuous functions from $A$ to $Y$ that converges to a function $f : A \to Y$ uniformly on $A$. Given any $x \in A$, prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$ for every sequence $\{x_n\}$ in $A$ with $x_n \to x$.

5. Compute and plot the Bernstein polynomials for $1/(1 + x)$ on $[0, 1]$ of degrees $1, 2, 3$.

6. Is the set of functions $\left\{ x^2 + \dfrac{n}{1 + n}\, x \right\}_{n=1}^{\infty}$ taking $\mathbb{R}$ to $\mathbb{R}$ (with the usual metric) equicontinuous on $[0, 1]$?

7. Assume that $\{f_n\}$ is a sequence of functions with $f_n : [a, b] \to \mathbb{R}$ for all $n$, where we take the usual metric on $[a, b]$ and $\mathbb{R}$.

   (a) Assume that $\{f_n\}$ is uniformly Lipschitz continuous on $[a, b]$ with constant $L$ (See Example 7.4.7). Prove that $\{f_n\}$ is equicontinuous on $[a, b]$.

   (b) Assume that $\{f_n\}$ is uniformly Lipschitz continuous on $[a, b]$ with constant $L$ and further that $\{f_n(x)\}$ is a uniformly bounded sequence of numbers for some $a \leq x \leq b$. (1) Prove that $\{f_n\}$ has a subsequence that converges uniformly on $[a, b]$. (2) Prove the second assumption is needed with an example.

   (c) Assume that $\{f_n\}$ is a bounded subset of $C([a, b])$. Show that the sequence of functions $\{F_n\}$ from $[a, b]$ into $\mathbb{R}$ defined by
   $$F_n(x) = \int_a^x f_n(s)\, ds$$
   contains a subsequence that converges uniformly on $[a, b]$. (This is a remarkable result, consider $\{\sin(nx)\}$ for example.)

8. Suppose that $\{f_n\}$ is a sequence of functions from $[a, b]$ into $\mathbb{R}$ (with the usual metric on $[a, b]$ and $\mathbb{R}$) that is equicontinuous on $[a, b]$ and converges pointwise on $[a, b]$. Prove that $\{f_n\}$ converges uniformly on $[a, b]$.