Math 517, Assignment 4
Due Friday, October 31
Remember to provide full reasoning for all answers!

1. Using the definition, show that $\lim_{x\to 0} 1/x$ does not exist.

2. (a) Using the $\delta - \epsilon$ formulation, show that $\sqrt{x}$ is continuous on $[0, \infty)$. *Hint:* Treat $x = 0$ as a separate case.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1^2 x_2}{x_1^4 + x_2^2}, & (x_1, x_2) \neq (0,0), \\ 0, & (x_1, x_2) = (0,0). \end{cases}$$

   Show that $\lim_{x\to 0} f(x, mx) = 0$ for every $m \in \mathbb{R}$ but f is discontinuous at $(0,0)$. Interpret this result.

4. Let $C^1([a,b])$ be the metric space of continuously differentiable functions on $[a,b]$ with metric

$$d(f,g) = \sup_{a\leq x\leq b} |f(x) - g(x)| + \sup_{a\leq x\leq b} |f'(x) - g'(x)|.$$

   Let $D : C^1([a,b]) \to C([a,b])$ be defined by $D(f) = f'$. Prove that D is continuous.

5. Let $C([a,b])$ be the metric space of continuous functions on $[a,b]$ with metric

$$d(f,g) = \int_a^b |f(t) - g(t)|\, dt.$$

   Define $T : C([a,b]) \to \mathbb{R}$ be

$$T(f) = \int_a^b f(t)\, dt.$$

   Is T continuous?

6. Let A be a set and x a point in a metric space (X, d). Define the "distance from x to A" to be

$$d(x, A) = \inf_{y\in A} d(x, y).$$

   (a) Show that for fixed A, the function $f : X \to \mathbb{R}$ (where we take the usual metric on $\mathbb{R}$) given by $f(x) = d(x, A)$ is continuous.
   (b) Show that $\{x : d(x, A) = 0\} = \overline{A}$.

7. Let f and g be continuous maps of a metric space X into a metric space Y and let E be a dense subset of X. (a) Prove that $f(E)$ is dense in $f(X)$. (b) If $g(x) = f(x)$ for all x in a dense subset E of X, prove that $g(x) = f(x)$ for all x in X. (c) Explain why these results are important for our everyday experience with functions and numbers.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that is periodic, i.e., there is a number T such that $f(x+T) = f(x)$ for all x in $\mathbb{R}$. (a) Show that f has finite maximum and minimum values on $\mathbb{R}$. (b) Show that f is uniformly continuous on $\mathbb{R}$.

9. (a) Let X be a metric space and suppose that f is uniformly continuous on the compact subsets $E_1, \cdots, E_n$ of X. Prove that f is uniformly continuous on $\cup_{i=1}^n E_i$.
   (b) Give an example of a function f is that continuous on $\mathbb{R}$ and a sequence of compact intervals $E_1, E_2, \cdots$ on each of which f is uniformly continuous, yet f is <u>not</u> uniformly continuous on $\cup_{i=1}^\infty E_i$.