Math 517, Assignment 3
Due Friday, October 17
Remember to provide full reasoning for all answers!

1. A collection $\{G_\alpha\}$ of open subsets of a metric space (X, d) is called a base for X if the following holds: for every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in G_\alpha \subset G$ for some α. This says that every open set in X is the union of a sub-collection in $\{G_\alpha\}$. Prove that every separable metric space has a *countable* base. (Hint: take all neighborhoods with rational radius and center in some countable dense subset of X.)

2. Prove that if a subsequence of a Cauchy sequence in a metric space converges, then the full sequence converges to the same limit.

3. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d). Show that $\{d(x_n, y_n)\}$ converges whether or not $\{x_n\}$ and $\{y_n\}$ converge.

4. Let (X_1, d_1) and (X_2, d_2) be two complete metric spaces. Is the product space $X_1 \times X_2$ complete?

5. Show that any closed ball in $\mathcal{M}([a, b])$ is <u>not</u> compact.

6. Prove the following Theorem: *A metric space X is complete if and only if the intersection of every descending sequence of closed balls whose radii approach zero consists of a single point.*

   Hint: If X is complete, use the sequence of closed balls to construct a Cauchy sequence. For the other direction, if $\{x_n\}$ is a Cauchy sequence, first show there is a subsequence $\{x_{n_k}\}$ with the properties

   - $d(x_{n_{k+1}}, x_{n_k}) < 2^{-k}$ *for* $k = 1, 2, 3, \cdots$
   - $\overline{N_1(x_{n_1})} \supset \overline{N_{1/2}(x_{n_2})} \supset \cdots \overline{N_{1/2^{k-1}}(x_{n_k})}$, *where* $\overline{N_r(x)} = \{y \in X : d(y, x) \leq r\}$

7. For any real sequence $\{x_n\}$, define $s_n = (x_1 + x_2 + \cdots + x_n)/n$. Prove that

$$\limsup_n s_n \leq \limsup_n x_n$$

   and give an example that shows the inequality can be strict.