Math 517, Assignment 2
Due Monday, September 29
Remember to provide full reasoning for all answers!

1. Show that every point of a set $A \subset X$, where (X, d) is a metric space, is either an isolated point or a limit point.

2. Consider the convergence of sequences $\{x_n\} = \{x_{n,1}, x_{n,2}, x_{n,3}, \cdots\}$ to a point $y = \{y_1, y_2, y_3, \cdots\}$ in ℓ_2.

   (a) Show that if $x_n \to y$, then $x_{n,i} \to y_i$ for $i = 1, 2, 3, \cdots$. But, construct an example that shows that $x_{n,i} \to y_i$ for $i = 1, 2, 3, \cdots$ does not imply that $x_n \to y$.

   (b) Consider the subspace of ℓ_2,

   $$H = \{x = \{x_1, x_2, x_3, \cdots\} : x \in \ell_2 \text{ and } |x_i| \le 1/i,\ i = 1, 2, 3, \cdots\},$$

   called the Hilbert Cube. Show that a sequence $\{x_n\} = \{x_{n,1}, x_{n,2}, x_{n,3}, \cdots\}$ in H converges to a point $y = \{y_1, y_2, y_3, \cdots\}$ if and only if $x_{n,i}$ converges to y_i for $i = 1, 2, 3, \cdots$.

3. Let $f : A \subset \mathbb{R} \to \mathbb{R}$ have the property: for every $x \in A$ there is an $\epsilon > 0$ such that

   $$f(t) > \epsilon \text{ if } t \in (x - \epsilon, x + \epsilon) \cap A.$$

   (We might say that f is "locally bounded away from zero on A".) Show that if the set A is compact, then there is some $c > 0$ such that

   $$f(x) > c \text{ for all } x \in A.$$

4. A function $f : \mathbb{R} \to \mathbb{R}$ is locally increasing at a point x if there is a $\delta > 0$ such that

   $$f(s) < f(x) < f(t)$$

   whenever

   $$x - \delta < s < x < t < x + \delta.$$

   Show that a function that is locally increasing at every point in $\mathbb{R}$ must be increasing, i.e., $f(x) < f(y)$ for all $x < y$.

5. Let (X, d) be a metric space.

   (a) Let $\{K_1, K_2, \cdots, K_n\}$ be a set of compact subsets of X. Prove that $\cup_{i=1}^{n} K_i$ is a compact subset of X.

   (b) Let $\{K_\alpha\}$ be a set of compact subsets of X. Prove that $\cap_\alpha K_\alpha$ is a compact subset of X.

6. Let (X, d) be a metric space with the discrete metric d (Definition 2.2.3). What subsets of X are compact?

7. Describe an open cover of the open unit square $\{(x, y) : 0 < x < 1,\ 0 < y < 1\}$ that has no finite subcover.

8. Let $\{x_n\}$ be a sequence in a metric space (X, d) that converges to a limit x. Show the set $\{x, x_1, x_2, \cdots\}$ is compact.

9. Show that the product of two compact metric spaces furnished with the product metric (Problem 6, Assignment 1) is a compact metric space.