Educ Stud Math
DOI 10.1007/s10649-016-9737-0

A characterization of a unified notion of mathematical function: the case of high school function and linear transformation

Michelle Zandieh[1] · **Jessica Ellis**[2] · **Chris Rasmussen**[3]

Abstract As part of a larger study of student understanding of concepts in linear algebra, we interviewed 10 university linear algebra students as to their conceptions of functions from high school algebra and linear transformation from their study of linear algebra. An overarching goal of this study was to examine how linear algebra students see linear transformation as related to high school function. Analysis of these data led to a characterization of student responses into three categories of mathematical structures used to discuss function: properties, computations, and a series of five interrelated clusters of metaphorical expressions. In this paper, we use this analytic framing for exploring the question: to what extent does each of the students in this study have a unified concept image of function across two mathematical contexts, high school algebra and their study of linear algebra? We found that students who expressed a unified notion of function used metaphorical language to bridge any gaps between the notion of function from high school and the notion of linear transformation from linear algebra. We conjecture that the framing of computations, properties, and metaphorical clusters could be extended to discussions of functions in contexts with other mathematical domains. Future research that further examines the extent to which undergraduate students develop a unified concept image of function could then lead to various design research efforts aimed at explicitly fostering such a unified understanding.

Keywords Concept image · Conceptual metaphor · Function · Linear transformation · One-to-one · Linear algebra

✉ Michelle Zandieh
zandieh@asu.edu

[1] Science and Mathematics Faculty, College of Integrative Sciences and Arts, Arizona State University, Polytechnic Campus, Mesa, AZ 85212, USA

[2] Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1874, USA

[3] Department of Mathematics and Statistics, San Diego State University, San Diego, CA 92182-7720, USA

Published online: 05 December 2016

As part of a larger study of student understanding of concepts in linear algebra, we interviewed 10 university linear algebra students as to their conceptions of functions from high school algebra and linear transformation from their study of linear algebra. An overarching goal of this study was to examine how linear algebra students see linear transformation as related to high school function. Analysis of these data led to a characterization of student responses into three categories of mathematical structures used to discuss function: properties, computations and a series of five interrelated clusters of metaphorical expressions. In this paper, we use this analytic framing for exploring the question: to what extent does each of the students in this study have a unified concept image of function across two mathematical contexts, high school algebra and their study of linear algebra? We found that students who expressed a unified notion of function used metaphorical language to bridge any gaps between the notion of function from high school and the notion of linear transformation from linear algebra. We conjecture that the framing of computations, properties, and metaphorical clusters could be extended to discussions of functions in contexts with other mathematical domains. Future research that further examines the extent to which undergraduate students develop a unified concept image of function could then lead to various design research efforts aimed at explicitly fostering such a unified understanding.

The function concept is a pervasive idea in mathematics, but one that remains problematic for many students (Carlson, 1998; Harel & Dubinsky, 1992; Oehrtman, Carlson, & Thompson, 2008). From calculus to linear algebra to differential equations to abstract algebra, undergraduate students encounter the notion of function again and again, but these encounters may or may not be occasions for them to renew and deepen their understanding of function. In the USA, linear algebra students deal extensively with functions in the form of linear transformations, including their inverses, and properties such as one-to-one and onto, but it is not clear to what extent such encounters draw on and extend how they think about function. In the USA and elsewhere, an explicit goal of mathematics education is to support students in not only learning multiple fields of mathematics, but to make connections between these fields and recognize when ideas from one content area translate to another (Gowers, Barrow-Green, & Leader, 2010; National Council of Teachers of Mathematics, 2000). In this report, we analyze individual problem solving interviews with 10 linear algebra students to investigate the extent to which they view function in the context of high school algebra as similar to or different from function in the context of linear algebra. We further examine how students see one-to-one functions and one-to-one linear transformations as mathematically connected, both in the interest of how students broadly understand the notion of one-to-one, and as informative of how students understand mathematical function itself, regardless of specific content areas.

The nature of students' conceptions of function has a long history in the mathematics education research literature. This work includes Monk's (1992) pointwise versus across-time distinction, the APOS (action, process, object, scheme) view of function (e.g., Breidenbach, Dubinsky, Hawkes, & Nichols, 1992; Dubinsky & McDonald, 2001), and Sfard's (1991, 1992) structural and operational conceptions of function. A comparison of these views may be found within Zandieh (2000). More recent work has focused on understanding of function with covariational reasoning (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, 1995). A summary with a focus towards covariational reasoning is found in Oehrtman et al. (2008). None of this research has looked at the concept of function in linear algebra, however. There have been a few studies on student understanding of linear transformation (Dreyfus, Hillel, & Sierpinska, 1998; Portnoy, Grundmeier, & Graham, 2006), and none focused on the functional nature of the linear transformations. Our study thus contributes to literature in this under researched but highly significant area.

1 Theoretical framing

Many past studies of student understanding of a mathematical feature have drawn on the seminal work of Vinner and colleagues and their development of the constructs of concept image and concept definition (e.g., Artigue, 1992; Rasmussen, 2001; Wawro, Sweeney, & Rabin, 2011; Wilson, 1993; Zandieh, 2000; Zandieh & Rasmussen, 2010). In the analysis presented here, we use the notion of concept image as a starting point and then extend this framing to include more recent approaches related to conceptual metaphors. In so doing, we offer a way to bring together two theoretical lenses that in past have been used in isolation.

The term concept image has been used to refer to the "set of all mental pictures associated in the students' mind with the concept name, together with all the properties characterizing them"; (Vinner & Dreyfus, 1989, p. 356). Tall and Vinner (1981) describe a person's concept image for a particular concept as "the total cognitive structure that is associated with the concept."; A person's *evoked concept image* is the portion of the concept image that is activated at a given time. In this study, we attend to the aspects of student's concept images of function and linear transformation that are evoked while filling out an in-class questionnaire and during a follow up interview.

Other studies contrast a student's use of concept image and *concept definition* (cf., Vinner, 1991). Tall and Vinner (1981) define the concept definition as "a form of words used to specify that concept"; (p. 152) and further delineate between a *personal* concept definition and a *formal* concept definition. A personal concept definition is the words an individual uses to explain a concept, whereas a formal concept definition is a concept definition that is accepted by the broader mathematical community. A personal concept definition may or may not match a definition acceptable by the standards of the broader mathematical community.

While students were not necessarily exposed to the identical formal concept definitions of function and one-to-one function in their high school algebra, precalculus, or calculus courses, we provide as a representative example of such definitions those found in Stewart (2007).

A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B (Stewart, 2007, p. 2).

A function f is called a one-to-one function if it never takes on the same value twice; that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ (Stewart, 2007, p. 148).

All students in this study were exposed to the following formal concept definitions of transformation and one-to-one transformation, coming from the text that was used in the course (Lay, 2006).

A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$ (Lay, 2006, p. 73).

A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$ (Lay, 2006, p. 87, emphasis in original).

A concept image framing allows us to group the associations a student has for a particular concept and set them in comparison with the group of associations a student has for another concept. In other words, we can set into contrast a student's concept image of function as typically encountered in high school with a student's concept image of linear transformation. We can also set up comparisons across students. Further, we also frame to what extent a student has a unified concept image of function that incorporates a larger set of ideas as being associated with the notion of function, as well as make sense of how students think about the connections between the two notions.

To delve into more detail about what is in a student's concept image, and to provide structure to the amorphous collection of things comprising one's concept image, we need a finer grained characterization of the variety of ways students may have of understanding a particular concept. For this purpose, we draw on studies that detail student concept images of mathematical notions using the construct of a *conceptual metaphor* (e.g., Lakoff & Núñez, 2000; Oehrtman, 2009; Zandieh & Knapp, 2006). Following from this work, our assessment is that a person's concept image of a particular mathematical idea will likely contain a number of conceptual metaphors as well as other cognitive structures.

In order to examine conceptual metaphors, we rely on metaphorical expressions. Lakoff and Johnson (1980) explain, "since metaphorical expressions in our language are tied to metaphorical concepts … we can use metaphorical linguistic expressions to study the nature of metaphorical concepts and to gain an understanding of the metaphorical nature of our activities"; (p. 456). Our work will describe clusters of metaphorical expressions that allow us first to characterize students' concept images of high school function and linear transformation. We then use the characterization to highlight the connections or discrepancies between a student's concept image of function and the student's conception of linear transformation, and to help us see how these conceptions are related to the student's expression of a unified concept image of mathematical function. By expressing a unified notion of a mathematical construct, we mean understanding various constructs as examples of the same phenomenon, regardless of differences in the specific contexts. Informed by the work surrounding concept image, conceptual metaphors, the research question we address in this study is: To what extent does each of the students in this study have a unified concept image of mathematical function? Answering this question will entail first a detailed analysis of students' concept images for function and for linear transformation, resulting in a characterization of students' concept images. This characterization is then leveraged to address the extent to which the students express a unified concept image of function.

2 Methods

The data for this report comes from an in-class questionnaire and semi-structured interviews with 10 students who were just completing an undergraduate linear algebra course in the USA. Consistent with most first post-secondary courses in linear algebra in the USA, this course included treatment of the majority of topics (e.g., span, linear independence, systems of linear equations, linear transformations, and eigentheory) recommended by the Linear Algebra Curriculum Study Group (Carlson, Johnson, Lay, & Porter, 1993). The instruction included a discussion of some connections between high school functions and linear transformations. In addition, the curriculum emphasized a geometric interpretation of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. (See Andrews-Larson, Wawro, & Zandieh, 2016, for a detailed discussion of the instruction in this class regarding linear transformations.) The class consisted of 30 students, most of whom had completed three semesters of calculus (at least two semesters were required). Approximately half had also completed a discrete mathematics course, most were in their second or third year of university, and student majors included computer engineering, computer science, mathematics, mathematics education, statistics, or another science or business field. Volunteers were requested for the interviews. Ten students volunteered, and we analyze their responses in this paper.

The interviews were videotaped and transcribed and student written work was collected. Students were given a questionnaire to fill out in class, and then, during the interview, they were asked semi-structured follow-up questions about their responses to the questionnaire. The focus of the interview was to obtain information about each student's concept image of function and each student's concept image of linear transformation and to see in what ways each student saw these as the same or different. To this end, we not only asked the students how they thought of a function or linear transformation, but also questions about characteristics that would be relevant to both functions and linear transformations, such as one-to-one, onto, and invertibility. For the purposes of this paper, we draw on student responses to the questions regarding definitions and examples of functions and linear transformations, and the concept of one-to-one. Students were asked four questions on the questionnaire and two Likert questions during the interview to this end, as shown in Table 1.

During the interview, students were first asked to read aloud their written responses to questions 1 and 2, and then asked question 5 with follow-up requests to explain their reasoning. They were then asked to review their responses to questions 3 and 4, and then asked question 6 again with follow up requests to explain their reasoning. Students' responses to these questions indicate to what extent they think of high school function and linear transformation belonging to a unified notion of a mathematical function, first in general and then specifically for a one-to-one function. All of the transformations that students were exposed to and worked (in both the class and the interview) with were linear transformations. As a consequence, when students were asked to talk about transformations in the interview, they necessarily talked about linear transformations. For this interview, the distinction between transformation and linear transformation was

Table 1 Questions from the in-class questionnaire and interview analyzed in this paper

Instrument	Question
In class questionnaire	1. In the context of high school algebra, explain in your own words what a function is.
	2. In the context of linear algebra, explain in your own words what a transformation is.
	3. (a) In the context of high school algebra, give an example of a function that is 1-1. Explain how you know that it is 1-1.
	3. (b) In the context of high school algebra, give an example of a function that is not 1-1. Explain how you know that it is not 1-1.
	4. (a) In the context of linear algebra, give an example of a linear transformation that is 1-1. Explain how you know that it is 1-1.
	4. (b) In the context of linear algebra, give an example of a linear transformation that is not 1-1. Explain how you know that it is not 1-1.
Likert interview questions	5. Please indicate, on a scale from 1 to 5, to what extent you agree with the following statement: "A linear transformation is a type of function."
	6. Please indicate, on a scale from 1 to 5, to what extent you agree with the following statement: "1-1 means the same thing in the context of functions and the context of linear transformations."
Example follow up questions during interview	Tell me more about why you strongly agree.
	So those two things sound pretty different on the surface at least; can you say more about the way in which it might be similar for the two?

not emphasized. Note that in the US context, we expected that referring to "the context of high school algebra"; would prompt students to recall their understanding of function from algebra, precalculus, and calculus courses (e.g., polynomial functions, log and exponential functions, and trigonometric functions). In fact those are the kind of functions students referred to when they were asked this question.

Data analysis proceeded along the lines of grounded theory as described by Glaser and Strauss (1967). In this approach, conjectures are formed post hoc by trying to identify patterns and regularities in the data and then conjecturing about the nature and character of these ideas. Grounded theory was used to analyze students' responses to questions 1–4 as well as their elaborations during the interview on their responses to questions 5 and 6. After students responded to questions 5 and 6, the interviewer followed up with questions to gain more insight into their thinking, such as "Tell me more about why you strongly agree"; or "So those two things sound pretty different on the surface at least; can you say more about the way in which it might be similar for the two?"; The data was analyzed following the constant comparison method and the final coding was negotiated between the authors of this paper. We found student responses to the Likert questions (5 and 6) to be less useful than their responses to the follow up questions that accompanied these responses and so we did not incorporate the Likert ratings as part of the analysis presented in this paper. Nonetheless, the Likert questions were important for setting up the richness of student responses to the follow up questions.

3 Results

3.1 Characterization of students' concept images

In order to understand the extent to which students expressed a unified concept image of a mathematical function, we first needed a way to characterize students' concept images of function and their concept image of linear transformation. Through this characterization, we found that there were three main components of students' concept images of these ideas: *computations*, *properties*, and *clusters of metaphorical expressions*. The computation component has no subcategories while the other two components have multiple subcategories. In this section, we provide illustrative examples of student reasoning for each of these ways of thinking about mathematical function.

Computations (Comp) While responding to the interview tasks' follow-up prompts, some students drew upon language that describes a calculation or computation. We included both computational language that described how a function or linear transformation behaves and side computations done involving the function or transformation. For example, Randall used computational language (multiplication) to discuss how a linear transformation acts.

> *Randall:* A transformation is a multiplication of matrices that leads to a new image produced from the original matrix.

In the set of questions analyzed in this paper, computational language was used relatively infrequently. We saw this component utilized more prevalently when students were asked to reason about composition of functions and function inverses. For an in-depth analysis of this section of the interview, see Bagley, Rasmussen, and Zandieh (2015).

Properties The property category refers to a student statement that describes a characteristic of a function or transformation or their associated graph or matrix without describing the inner workings of the function or transformation. Descriptions of the inner workings of the function or transformation necessarily draw on terms from one or more of the clusters of metaphorical expressions, which we describe in more detail below. In Table 2, we illustrate examples of properties that were referenced in relation to function or linear transformation. The underlined portion of the student quote is the part of the utterance that is associated with the respective subcategory. In each of these examples, the students do not explain more in depth as to why, for example, a linearly independent transformation is one-to-one. Some of these properties, such as $P_{equations}$ and P_{LI}, were used by many students. Others, such as $P_{monotonic\ functions}$, were used by only one student.

In addition to drawing on properties and computations when reasoning about function and linear transformation, many students employed metaphorical language.

Clusters of metaphorical expressions We identified five different clusters of metaphorical expressions that students called upon when reasoning about function or linear transformation: input/ output, traveling, morphing, mapping, and machine. We next provide examples of each of these five metaphorical expressions and then illustrate ways in which students may combine metaphors by drawing on the underlying similar structure of the metaphors.

> Input/Output (IO). Input/output involves an input, which goes into something, and an output, which comes out. This can be viewed from the point of view of the person "putting in"; the input and "taking out"; the output, and/or from the point of view of the function or transformation "accepting,"; "receiving"; or "taking"; an input and "returning"; or "giving"; an output.
> Traveling (Tr). Traveling involves an entity in a beginning location that is sent or moved to an ending location. Some phrases that we found to be indicative of this cluster were the use of "gets sent,"; "goes to,"; "moving,"; "reach,"; "go back,"; and "get to."; These expressions were used almost exclusively when reasoning about linear transformation. We saw these expressions used to describe a pointwise change in location as well as a global move.

Table 2 Examples of select properties references by students

Code	Property	Example
$P_{equation}$	Reference to a function as an equation.	*Adam*: A function is an equation with a variable.
P_{map}[a]	Reference to a mapping relationship.	*Brad:* This is one output for every input.
$P_{shape\ of\ graph}$	Reference to the shape of a graph.	*Adam:* Because since it is a bowl shaped there are 2 *x*'s for every *y*.
$P_{monotonic\ function}$	Reference to strictly increasing or decreasing functions.	*Nila:* A 1-1 function would be a strictly increasing or strictly decreasing function because it is the only way that each *x* will have a unique *y* value and each *y* will have a unique *x* value.
P_{LI}/ P_{LD}	Reference to Linear (in)dependence (LI/ LD)	*Donna:* I said that was one-to-one because it's linear independent.

[a] This code is explained in more detail in the Map section

Morphing (Mor). Morphing involves a beginning state of an entity that changes or is morphed into an ending state of the same entity. There must be a clear sense that the beginning entity did not simply move to the new location (ending entity), nor was it replaced by the new output (ending entity), but that there was actually a metamorphosis of the beginning entity into the ending entity. Morphing may be used pointwise by imagining one object changing, or globally by imagining a collection of objects changing. We found the phrases "become,"; "transform,"; and "change"; to be indicative of this cluster.

Mapping (Map). Mapping involves two entities, and a relationship or correspondence between the two. This cluster is most closely related to the Dirichlet-Bourbaki definition of function, and was not commonly used by students when answering "explain in your own words what a function [linear transformation] is."; It was used more commonly when discussing one-to-one functions.

We found references to a "mapping,"; "rule,"; or "correspondence"; to be indicative of this cluster, as well as "per"; and "for";, as in "there is one input for/per every output."; The word "map"; used as a verb however, was used by students both related to and not related to the Mapping cluster. For instance, George said, "So if you have a dependent matrix, it could map multiple points, to maybe the same point, collapse dimension or do weird stuff."; This use of map is indicative of the Mapping cluster.

On the other hand, on the questionnaire Gabe gave the example $[1 \quad 2 \;\; 24]$ and wrote "Not one-to-one. Dependent vectors will not map into correct area."; This was labeled as being associated with the traveling cluster since there is indication of movement to a location "into correct area.";

Since this cluster is the one most closely associated with the formal definition of one-to-one stated in textbooks, we struggled with whether a student's utterance should be labeled as Map when they stated something that could have been a memorized definition. We therefore have chosen to use P_{map} to indicate a statement that expresses a mapping relationship, but to reserve Map for when there is some argument made beyond just stating an example and a phrase similar to the formal definition.

Machine (Mach). Expressions in the machine cluster include a beginning entity or state, an ending entity or state, and a reference to a tool, machine or device *that causes* the entity to change from the beginning entity/state into the ending entity/state. A necessary component to expressions in this cluster is language that indicates that the function or transformation is performing the action on the entity. We found the phrases "acts on"; and "produces"; to be indicative of this cluster.

As shown in Table 3, these five clusters share a common structure: each involves an entity 1, an entity 2, and a description about how these two are connected. (Note that not all three parts of a structure must be stated by a student for the statement to be classified as part of a particular cluster.) This common structure allows students to combine expressions in complex ways.

Combined expressions. There are several general things to note when comparing across the clusters of metaphorical expressions for function and linear transformation. Each of the clusters has the same general structure, and they are often used in combination in student reasoning. In particular, since input/output focuses more on the beginning and ending entity, it is the easiest cluster to combine with other clusters. However, students often flow from one cluster to another, even in the same sentence. This may be indicative of students' use of *conceptual blending* with these clusters of metaphorical expressions. Conceptual blending (Fauconnier & Turner, 2002) refers to two cognitive structures with

Table 3 Structure of metaphorical expressions

Cluster	Entity 1	Middle	Entity 2	Example
Input/Output (IO)	Input(s)	Entity 1 goes/is put into something and entity 2 comes/is gotten out.	Output(s)	A function $f(x) = y$ means that putting x inside would give you a specific output, y.
Traveling (Tr)	Beginning location(s)	Entity 1 is in a location and moves into a (new) location where it is called entity 2.	Ending location(s)	A transformation is moving a point or object in a certain direction
Morphing (Mor)	Beginning state of the entity(ies)	Entity 1 changes into entity 2.	Ending state of the entity(ies)	So if you have a dependent matrix, it could map multiple points, to maybe the same point, collapse dimension or do weird stuff.
Mapping (Map)	First entity	Entity 1 and entity 2 are connected or described as being connected by a mapping (a description of which First entities are connected to which Second entities).	Second entity	So if you have a dependent matrix, it could map multiple points, to maybe the same point, collapse dimension or do weird stuff.
Machine (Mach)	Entity(ies) to be processed	Machine, tool, device acts on entity 1 to get entity 2.	Entity after being processed	Pretty much anything you toss in here, this is still that ktransformation should be able to act on it.

a correspondence between them. This correspondence may be given by a metaphor, and in that case the correspondence is called a *metaphorical blend*. Conceptual blends, both non-metaphorical and metaphorical blends, are thought to the essence of the understanding of abstract mathematics (Lakoff & Núñez, 2000). For example, Lawson combines the mapping, machine and input/output.

Lawson: Because it essentially does the same thing. So it's like, how I have here a rule that assigns, essentially a function is the same thing, you put in an input, and it manipulates that input and turns it into an output.

The example of Lawson illustrates the complex ways in which students draw upon metaphorical blends within their concept images of function and linear transformation. We found that, more often than not, students drew on metaphorical blends combined with properties and computations, rather than drawing on one component alone.

3.2 Student concept images of function and linear transformation

In this section, we use student responses to questions 1–4 to illustrate examples from students' concept images of high school function, linear transformation, one-to-one function and one-to-one linear transformation. We note that language referring to particular properties and metaphorical clusters occur more often when talking about function versus linear transformation.

Questions 1 and 2 asked students to describe what function and linear transformation are, while questions 3 and 4 asked students to give examples of functions and linear transformations that are and are not one-to-one, and to explain why or why not. Table 4 gives each student's

Table 4 The components of students' evoked concept images as stated on the written questionnaire

Student	High school function (Q1)	Linear transformation (Q2)	One-to-one function (Q3)	One-t- one linear transformation (Q4)
Adam	$P_{equation}$	Tr	P_{map}, $P_{shape\ of\ graph}$	P_{LI}, Map
Brad	IO, Mor	IO, Mor	IO, P_{map}	IO, P_{map}, Tr
Donna	IO	Mor	P_{map}, IO, P_{HLT}, P_{VLT}, $P_{shape\ of\ graph}$	P_{LI}, P_{map} IO, P_{line}, P_{LD} $P_{infinite}$
Gabe	$P_{equation}$, IO	$P_{equation}$, Tr	IO, Map	P_{LI}, P_{square}, Tr, P_{LD}
Jerry	IO	Mach	–	–
Josh	Comp	Mor, Mach	P_{HLT}	P_{LI}, $P_{invertible}$, $P_{diagonal}$, P_{LD}
Lawson	IO	Map, IO	P_{map}, Map	P_{map}
Nigel	Mach	Mor	P_{HLT}	P_{LI}, $P_{invertible}$, P_{LD}, Morph, P_{line}
Nila	$P_{equation}$, IO	Mach	$P_{monotonic\ function}$, P_{map}, $P_{even\ function}$	P_{LI}, IO, P_{map}, P_{LD}, P_{line}
Randall	$P_{equation}$, IO, Map, P_{VLT}	Comp, Mor	Map	P_{map}

evoked concept image(s) for questions 1–4. Student responses for each question often involved more than one utterance, each of which was categorized as computational, properties, or metaphoric expressions. Table 4 shows all of the coded utterances for each student per question. The order of the coding reflects the temporal order in which the utterances occurred. As shown in Table 4, students more often rely on metaphorical components when answering questions 1 and 2, and more often properties when answering questions 3 and 4.

By comparing each student's responses to these questions, we can see that certain clusters of metaphorical expressions are called upon more frequently than others when reasoning about function or linear transformations. For instance, when discussing function for question 1, the input/output cluster (7 students) and the property of being an equation (4 students) were the most prevalent. By contrast when answering question 2 about linear transformation, the morphing cluster (5 students) and the machine cluster (3 students) were most common. The traveling cluster was not used by students describing functions in question 1, but was used in question 2 for linear transformations (2 students). Notice also that all but one of the students described function using different clusters of expressions than they used for linear transformation. There was likely more variety in students language around transformation because the subject was newer to them than function and also because transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ lend themselves to imagery of vectors or sets of vectors moving or change into something else. By contrast, imagery of functions from $\mathbb{R}$ to $\mathbb{R}$ does not typically focus on a number in $\mathbb{R}$ being changed or moved but rather on the set of (input, output) ordered pairs that make up a graph.

For questions 3 and 4, students were asked to first give examples of functions and linear transformations that are and are not one-to-one, and then to explain why or why not. Typical examples provided were similar to those given by Brent, who gave the examples of $f(x) = x$, $f(x) = x^2$, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ for a one-to-one function, a not one-to-one function, a one-to-one linear transformation, and a not one-to-one linear transformation, respectively.

Only three students did not give an explicit example for a high school function on the interview questionnaire. Two of these students (Donna and Josh) stated on the questionnaire that it must pass the horizontal line test and at some point in the interview discussion gave an example of a straight line for one-to-one and a parabola for not one-to-one. A third student (Nila) gave descriptions of classes of functions on the questionnaire such as "strictly increasing"; for one-to-one and "a parabola or hyperbola"; for not one-to-one.

For the examples of linear transformations that are one-to-one or not one-to-one, six students gave correct examples similar to Brent's; two students (Josh and Nila) did not give an example but described that the column vectors must be linearly independent for 4a and linearly dependent for 4b. Thus, the majority of these students were able to identify basic functions and linear transformations that were and were not one-to-one. This shows a basic level of understanding of these mathematical constructs.

For questions 3 and 4, students primarily used properties to describe why their examples were or were not examples of a one-to-one function or a one-to-one linear transformation. More specifically, students primarily referred to P_{map} (and/or Map, IO) (7 students), P_{HLT} (3 students), and $P_{shape\ of\ graph}$ (2 students, plus Nila's references to monotone and even functions) for question 3. In contrast in answer to question 4, students primarily referred to P_{LI} (and/or P_{LD}) (6 students) P_{map} (and/or Map, IO) (5 students) with at least two students mentioning each of Tr, P_{line}, and $P_{invertible}$.

3.3 Toward a unified concept image of a mathematical function

In this section, we examine the extent to which students expressed a unified concept image of function. By expressing a unified notion of a mathematical construct, we mean understanding various constructs as examples of the same phenomenon, regardless of differences in the specific contexts. For the purpose of this paper, we wanted to know if linear algebra students understood both high school functions and linear transformations as examples of mathematical function, and if students considered the notion of one-to-one to mean the same thing in both contexts. To determine this, we analyzed student responses, not only to the Likert questions (Q5 and Q6), but also to their explanation of why or how they saw the two contexts as similar or different.

In answer to question 5, all ten students agreed that a linear transformation is a type of function. In addition, all but two students, when asked, were able to correctly describe both a function and a linear transformation using a cognitive structure that was the same across two contexts. Students used an input-output (IO), machine or morphing metaphor, for comparison across the two. As an example, Lawson used all three in the excerpt above. While all ten students agreed that a linear transformation is a type of function on Q5, only six of the students agreed (with a 4 or a 5) that one-to-one means the same thing in the two contexts (Q6). Each of the three students that circled 3 made a comment similar to Josh's explanation for why he chose 3: "I'm going to say neutral on that, but I know it's true, I just don't know how to explain it, that's why I picked neutral!";

In responses to Q6, only three students were able to fully and correctly explain one-to-one in both contexts using the same cognitive structure. Vignettes 3 and 4 below are examples of students in this category. Five students, who had correctly explained function across the two contexts, struggled to explain one-to-one across the two context. An example of such a student is in Vignette 2. Two students struggled to explain a cognitive structure for both function and one-to-one function, Donna and Josh. Vignette 1 tells Donna's story.

4 Vignettes

The first vignette comes from Donna, as a representative of the first group of students. Donna was neutral on Question 6 (Likert choice 3) and when we analyzed the components of her concept images for function and linear transformation when answering question 6, we found that she relied heavily on properties. The second vignette comes from Adam, as a representative of the second group of students. Adam strongly agreed that one-to-one means the same thing in the two contexts (Likert choice 5) but when we analyzed the components of his concept images for function and linear transformation when answering question 6, we found that he relied both on properties and metaphorical reasoning. We present two vignettes from the third group: one from a student, Nigel, who started out the interview unsure about why one-to-one means the same thing in the two contexts, but by the end of the interview convinced himself strongly of their relationship; the last vignette comes from Brad, who strongly agreed with Question 6 (Likert choice 5) and whose explanation relied solely on metaphorical components.

4.1 Vignette 1—reliance on properties

Donna's initial description of one-to-one in the context of function mentioned a mapping understanding (P_{map}), but she relied on the vertical and horizontal line test to determine when a function is or is not one-to-one (P_{VLT}, P_{HLT}, $P_{shape\ of\ graph}$). In the context of linear transformation she briefly mentioned a mapping understanding (P_{map}), but instead relied primarily on linear (in)dependence (P_{LI}, P_{LD}). When asked if she was thinking of one-to-one the same way in both descriptions, Donna replied that she was not:

> *Donna*: No, I don't think so, because I was thinking in terms of just simple, what I learned in high school, how the one-to-one function is something that has exactly one input for one output. And then in linear algebra, I was thinking in terms of linear dependency and independency and what we had learned prior in the class.

Donna's reliance on properties about one-to-one functions and one-to-one linear transformations did not enable her to reconcile these notions. This vignette further indicates that Donna, and students classified in the same grouping, did not express an evoked concept image of a unified mathematical (one-to-one) function.

4.2 Vignette 2—struggling to explain one-to-one across the two contexts

Adam's initial description of one-to-one used the shape of the graph and brief mapping language for function (P_{map}, $P_{shape\ of\ graph}$), and linear independence and stronger mapping language for linear transformation (P_{LI}, Map). He strongly agreed that one-to-one meant the same thing in both contexts, and explained that he strongly agreed because "it just feels like the same thing, like if you put one in, it only comes out as one possibility… it's hard to explain.";

The interviewer then pointed out that the way he described one-to-one in both contexts "looked kind of different"; and asked him to elaborate how he thought about them as the same. Adam then described how he saw linear (in)dependence related to one-to-one, drawing on input/output and traveling language. The interviewer asked him to elaborate examples of function and transformation that are not one-to-one, which he did by referencing properties for both, the mapping metaphor for functions, and computational language for linear transformation. When the interviewer prompted Adam to highlight how these two explanations were compatible, he realized that they were not as compatible as he had originally thought, concluding that his understandings of one-to-one in the two contexts are "a little different";:

> *I:* Because I was interested in this on the one hand, the answers look kind of different, right? Because you're describing something about the x and y's here in part 3. But in part 4, you're saying stuff about linear independent or something like that. So on the surface, your answers look kind of different. But I was wondering how you thought about them as being similar?
>
> *Adam*: Because if its linear independent you could span the entire range or whatever. And I just feel like if you can span everything when you do a transformation to it, it won't, it will only go to one place. We had transformations that were linear dependent, so we ended up going to a line. Any point you put in went to that line, it would transform $y = 3x$ or something. So I just felt like, if you had linear independent ones, it wouldn't go anywhere other than that one point. Every input only goes to one output.

I: Why don't you say just a tad more about why y is equal to x squared is not one-to-one?
Adam: I was just thinking of it was the horizontal line test, where if y equals 1 or 2, you could have two possibilities. Because if it comes out like a bowl, it goes up [graphs a parabola]. So if your y equals 1, it goes 1, there to there. Two possibilities of your x.
I: And do you have a similar example for a function, a linear transformation that's not one-to-one?
Adam: I would guess, that looks like $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ because they're the multiples of each other, it would end up on the line of $y = x$, like everything would end up there, if you multiply out [draws a horizontal line through $y = x^2$] whatever you put in.
I: Does it have a similar storyline like this one [points to the parabola], like how you were telling me there's one y value, there's two x values that go to it?
Adam: Not really, I don't know.
I: It's maybe a little different?
Adam: Yeah, a little different.

This vignette exhibits a case where the student believes one-to-one function is a broad mathematical concept that encompasses one-to-one high school function and one-to-one linear transformations, but his concept images of one-to-one high school function and one-to-one linear transformations hinder him from more fully having a unified concept image for one-to-one function. His descriptions of one-to-one in each mathematical context did not completely recognize the underlying structure of one-to-one function.

4.3 Vignette 3—moving toward a unified concept image

When asked question 3, Nigel circled 4 and said, "I mean, I would agree, but at the same time, I don't really see a solid connection that I can explain myself."; His initial explanation of one-to-one for function relied exclusively on the horizontal line test (P_{HLT}), and linear (in)dependence for one-to-one linear transformations (P_{LI}, P_{LD}). When explaining why he agreed to Question 6, Nigel incorporated metaphorical language into his reasoning about one-to-one linear transformations, specifically morphing ("transformed";) and traveling language ("to that spot";):

Nigel: I've learned of one-to-one means the horizontal line test. … But for linear transformations, I see it as, here's this vector, if it gets transformed by a one-to-one transformation, it's going to get plotted to its own specific new vector, and no other vector will be transformed to that spot.

When asked to reconcile his understandings in the two contexts, the following exchange occurred.

I: So those two things sound pretty different on the surface at least, can you say more about the way in which it might be similar for the two?
Nigel: One-to-one is just for every y value, there's a unique x. … Maybe there's multiple y values for this x,… so $y = \sin(x)$, so it looks something like that. For this, if you have a horizontal line, you just go down, you'll see across here there's … the same y values but for different x values. So, that's not one-to-one.

Nigel used mapping language to describe how the horizontal line test works. His language was more compatible with his description of one-to-one in the context of function, but he used different metaphors in the two contexts. He summarized his final understanding in the following way: "Say vector (−1, 1) gets transformed to (1, 1)… then no other vector is going to get transformed to (1, 1). So, I see it more with numbers [in the context of linear algebra]. Where with algebra in high school, I saw it more as a picture."; Nigel articulated the ways in which he recognized a parallel structure between one-to-one function and one-to-one linear transformation, but also identified the contextual differences that prevented him from entirely agreeing that the construct of one-to-one was the same in each context. In this interview, Nigel never completely reconciled his notions in terms of using identical language, but he did find some sense of understanding a compatibility that he had not recognized previously in that he said, "I never really explained it like that before!";

4.4 Vignette 4—a unified concept image

The fourth vignette illustrates the case of the two students (Brad, Randall) who exhibited a unified concept image of one-to-one function, and thus of mathematical function in general. Each of the two students in this category used similar language when describing examples in each context, indicating that their concept images of one-to-one function and one-to-one linear transformation were closely aligned. For example, Brad discussed a one-to-one function by saying, "This is one output for every input"; (IO, P_{map}). For a one-to-one transformation he said, "For every output there is one input to get there"; (IO, P_{map}, Tr). In each context Brad used brief language from the mapping cluster[1], i.e., for every ___ there is one ___. He also used input/output language in each context. A subtle difference was his inclusion of traveling language for linear transformation, with the phrase, "to get there."; Both Brad and Randall were able to give examples of functions that are not one-to-one in each context and to show that the function or linear transformation is not one-to-one by finding two input values that map to the same output value. Both Brad and Randall provided $f(x)=x$ as the one-to-one function and $f(x)=x^2$ as the non one-to-one function. Brad provided $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ as the one-to-one transformation, and Randall provided $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and both provided $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ as the non one-to-one function.

The ease with which Brad articulated the similar meaning of one-to-one in each context is indicative of his unified concept image of mathematical function, and specifically of one-to-one mathematical function. He highlighted this in his explanation of why one-to-one means the same thing in both contexts: "Just going back to the definition of it, what was that, at most one input for every output. So both, I mean the definition is satisfied whether you're using high school algebra or linear algebra."; Because Brad's concept image of one-to-one function is not tied specifically to the context of high school algebra or linear algebra, he can easily interpret the definition in either context.

4.5 Summary of the vignettes

This series of vignettes showcases a range in terms of student development toward a unified concept image of function. As shown in Vignette 1 and 2, we see that students who rely

[1] As noted above, Brad is coded here as P_{map} instead of Map because in his written work he does not make clear that he is not simply referencing a memorized relationship. In his verbal explanation he makes this distinction more clear.

primarily on properties about one-to-one in the two contexts had difficulties in identifying consistencies between one-to-one across the contexts. They focused on properties such as a “function is one-to-one if it passes the horizontal line test”; or “a linear transformation is one-to-one if the column vectors are linearly independent.”; In Vignette 3, we tendered an example of what it looks like for a student to be in transition towards a unified concept image of function. In contrast to Vignettes 1 and 2 and consistent with Vignette 3, the students illustrated in Vignette 4 drew on metaphorical language of one-to-one in the two mathematical contexts, and the compatibility of function across the two mathematical contexts was more clear to them. In summary, a reliance on properties appeared to impede students' development of a unified concept image of function while an ability to draw on metaphors facilitated such development.

5 Conclusion

In this paper, we illustrated an analytic framing for exploring whether a student has a unified concept image of function across two mathematical contexts? We used computations, properties and clusters of metaphorical expressions to delineate cognitive structures of students' concept images of function (and transformation). Each of the metaphorical clusters allows for the description of function and one-to-one function in terms of the relationship between what we refer to in Table 3 as “Entity 1”; and “Entity 2.”; The layering of the metaphorical clusters within one context and the compatibility of the metaphorical language across contexts allows for the recognition of similarities between the individual concept images of (one-to-one) function and (one-to-one) linear transformation. In contrast, when a student thinks in terms of a property such as the horizontal line test or the linear independence of column vectors in a matrix, then the connections across contexts are more difficult to make. The horizontal line test is specific to the context of a function from the real numbers to the real numbers, and the column vector test is specific to a linear transformation that is defined in terms of matrix multiplication. Either of these properties could be unpacked in terms of one or more of the metaphor clusters (including in terms of the formal definition of one-to-one, which fits in the mapping cluster). However, the tests in their most efficient, easy-to-apply format have been condensed or simplified in a way that hides the structural relationships that would allow students to understand the notion of function across the context of high school algebra and linear algebra.

From a pedagogical point of view, one goal of linear algebra is to expose students to mathematical functions beyond $\mathbb{R}$ to $\mathbb{R}$, and for students to make connections between the functions they dealt with in high school from $\mathbb{R}$ to $\mathbb{R}$ to the functions they deal with in linear algebra from $\mathbb{R}^n$ to $\mathbb{R}^m$, specifically those that can be written as a matrix times a vector. Students may have separate concept images for algebraic functions and linear transformations, but over time the connections between the conceptions should be forged to form a broader concept image of mathematical function. In this paper we investigated how students see similarities and differences across their concept image of “high school functions”; and their concept image of “linear transformations in my linear algebra class”;, and how they may reflect on these in ways that help them create a unified concept image of a mathematical function. This issue speaks to a broader goal of mathematics education: for students to be able to understand a construct, such as one-to-one function, across a number of different mathematical contexts.

One step to helping students develop these broader understandings is to identify how they understand the construct within various mathematical contexts. Our theoretical framing of concept image together with conceptual metaphor provides such a tool. With this tool, we were able to highlight similarities and differences across students' concept images of function and one-to-one function in each context. By making these comparisons, it became clear that a reliance on certain properties made developing a unified concept image of (one-to-one) function difficult. This phenomenon is likely to occur for other mathematical concepts as well. We see this research as an illustrative example toward exploring this larger phenomenon.

The main contribution of this paper is the creation of a frame in terms of computation, properties, and metaphorical clusters to characterize student understanding of function (e.g., transformation) that brings to light aspects not emphasized in other literature on student understanding of function and at the same time allows for a comparison across mathematical contexts in which functions may occur. In this paper, we showed that this characterization was useful in comparing student understanding of function across the mathematical contexts of (1) functions from $\mathbb{R}$ to $\mathbb{R}$ that might be studied in algebra, precalculus or calculus and (2) transformations from linear algebra. Functions are one of the main components of mathematics, regardless of the specific mathematical field, and are thus pervasive in undergraduate students' study of mathematics. Indeed, when describing the role of functions in mathematics, Gowers et al. (2010) state: "One of the most basic activities of mathematics is to take a mathematical object and transform it into another one"; (p. 10). Our theoretical framing of unified concept image of function relies on analyses of ten students from one linear algebra course. We conjecture that our characterization could be extended to discussions of functions in other contexts (such as in abstract algebra, differential equations, or complex analysis). A characterization of student understanding of function that can be used in many content areas allows for a description of a unified concept image of function—a concept image that considers high school functions from $\mathbb{R}$ to $\mathbb{R}$ and linear transformations to be part of the same larger concept image of function. Future research that further examines the extent to which undergraduate students develop a unified concept image of function could then lead to various design research efforts aimed at explicitly fostering such a unified understanding. We conjecture that this would be especially important for future secondary school teachers. Further, we envision the characterization of one's concept image in terms of properties, computations, and metaphors to be helpful in making sense of students' understandings across a wide range of mathematics, not just mathematical functions.

References

Andrews-Larson, C., Wawro, M., & Zandieh, M. (2016). *A hypothetical learning trajectory for conceptualizing matrices as linear transformations*. Manuscript submitted for publication.

Artigue, M. (1992). Cognitive difficulties and teaching practices. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 109–132). Washington, DC: The Mathematical Association of America.

Bagley, S., Rasmussen, C., & Zandieh, M. (2015). Inverse, composition, and identity: the case of function and linear transformation. *Journal of Mathematical Behavior, 37*, 36–47.

Breidenbach, D., Dubinsky, E., Hawkes, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247–285.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *CBMS issues in mathematics education: Research in Collegiate Mathematics Education* (Vol. 3, pp. 114–162). Providence: American Mathematical Society.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Carlson, D., Johnson, C., Lay, D., & Porter, A. D. (1993). The Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra. *The College Mathematics Journal, 24*(1), 41–46.
Dreyfus, T., Hillel, J., & Sierpinska, A. (1998). Cabri-based linear algebra: Transformations. *European Research in Mathematics Education I*, 209–221.
Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton, M. Artigue, U. Krichgraber, J. Hillel, M. Niss, & A. Schoenfeld (Eds.), *The teaching and learning of mathematics at university level: An ICMI Study* (pp. 273–280). Dordrecht: Kluwer.
Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York: Basic Books.
Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine Publishing Company.
Gowers, T., Barrow-Green, J., & Leader, I. (Eds.). (2010). *The Princeton companion to mathematics*. Princeton University Press.
Harel, G., & Dubinsky, E. (Eds.). (1992). *The concept of function: Aspects of epistemology and pedagogy. MAA Notes* (Vol. 25). Washington, DC: The Mathematical Association of America.
Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago: The University of Chicago Press.
Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.
Lay, D. C. (2006). *Linear algebra and its applications* (3rd ed.). Reading, MA: Addison-Wesley.
Monk, G. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 175–193). Washington, DC: The Mathematical Association of America.
National Council of Teachers of Mathematics (Ed.). (2000). *Principles and standards for school mathematics* (Vol. 1). Reston, VA: National Council of Teachers of Mathematics.
Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*, 396–426.
Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 150–171). Washington, DC: Mathematical Association of America.
Portnoy, N., Grundmeier, T. A., & Graham, K. J. (2006). Students' understanding of mathematical objects in the context of transformational geometry: Implications for constructing and understanding proofs. *Journal of Mathematical Behavior, 25*, 196–207.
Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*, 55–87.
Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36.
Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification – The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 59–84). Washington, DC: The Mathematical Association of America.
Stewart, J. (2007). *Essential calculus: Early transcendentals* (4th ed.). Belmont, CA: Thomson Higher Education.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.
Thompson, P. W. (1995). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education* (Vol. 1, pp. 21–44). Washington, DC: Conference Board on the Mathematical Sciences.
Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–79). Dordrecht, Netherlands: Kluwer.
Vinner, S., & Dreyfus, T. (1989). Images and definition for the concept of function. *Journal for Research in Mathematics Education, 20*, 356–366.
Wawro, M., Sweeney, G., & Rabin, J. M. (2011). Subspace in linear algebra: Investigating students' concept images and interactions with the formal definition. *Educational Studies in Mathematics, 78*, 1–19.
Wilson, M. R. (1993). *One preservice teacher's evolving understanding of mathematical functions*. Paper presented at the Annual Meeting of the American Educational Research Association, Atlanta, GA.
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *Research in Collegiate Mathematics Education* (Vol. 4, pp. 103–127). Washington, DC: Conference Board on the Mathematical Sciences.
Zandieh, M., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *Journal of Mathematical Behavior, 25*, 1–17.