VARIATIONAL CHARACTERIZATION OF A SUM OF CONSECUTIVE EIGENVALUES; GENERALIZATION OF INEQUALITIES OF PÓLYA-SCHIFFER AND WELY

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Abstract. Let \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \) be the eigenvalues of a vibrating system, an extremal property of \( \sum _i \lambda _i \) and \( \sum _i \lambda _{-1} \), suggested by the work of Pólya-Schiffer [1], is established and generalized to \( \sum _{k+1} \lambda _i \) and \( \sum _{k+1} \lambda _{-1} \): on the one hand in the sense of Poincaré, on the other in the sense of the “Max-Min” property of Courant-Weyl. We establish inequalities which reduce to those of Pólya-Schiffer [1] for \( k = 0 \) and to those of Weyl [2] for \( n = 1 \).

1. Definition of the “Rayleigh Trace” \( TR[L_n] \) on a linear space \( L_n \) and of the “Trace Inverse” \( TRinv[L_n] \)

We consider two positive definite quadratic forms \( A(\nu, \nu) \) and \( B(\nu, \nu) \) on a vector or functional space; the Rayleigh quotient will be \( R[\nu] = \frac{A(\nu, \nu)}{B(\nu, \nu)} \).

We will suppose that the beginning of the spectrum is discrete.

Given a linear subspace \( L_n \) of dimension \( n \), choose \( n \) vectors \( \nu_1, \ldots, \nu_n \) which are pairwise orthogonal in the metric \( B : B(\nu_1, \nu_j) = 0 \) if \( i \neq j \); we define

\[
(1) \quad TR[L_n] = R[\nu_1] + \cdots + R[\nu_n].
\]

This is the trace of the matrix associated with \( A \) in \( L_n \) under the metric \( B \): thus this definition is independent of the choice of \( \nu_1, \ldots, \nu_n \).

Now, choose \( n \) vectors \( \omega_1, \ldots, \omega_n \in L_n \) pairwise orthogonal in the metric \( A : A(\omega_i, \omega_j) = 0 \) if \( i \neq j \); we define

\[
(2) \quad TRinv[L_n] = \frac{1}{R[\omega_1]} + \cdots + \frac{1}{R[\omega_n]}.
\]

This is the trace of the matrix associated with \( B \) in \( L_n \) under the metric \( A \): thus this definition is independent of the choice of \( \omega_1, \ldots, \omega_n \).

2. Variational characterization of \( \sum _i \lambda _i \) and of \( \sum _i \lambda _{-1} \)

We part from the recurrent definition of the eigenvalues \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \) and of the corresponding eigenvectors \( u_1, u_2, u_3, \ldots \):

\[
\lambda_1 = \min _\nu R[\nu] = R[u_1]; \quad \lambda_2 = \min _{B(u_1, \nu) = 0} R[\nu] = R[u_2]; \quad \ldots
\]
For any \( n \geq 1 \),

\[
(3) \quad \lambda_1 + \lambda_2 + \cdots + \lambda_n = \min_{\text{choice of } L_n} TR[L_n].
\]

In effect, there exists in all \( L_n \):

a vector \( \nu_n \) which is \( B \)-orthogonal to \( u_1, \ldots, u_{n-1} \), so \( R[\nu_n] \geq \lambda_n \);

a vector \( \nu_{n-1} \) which is \( B \)-orthogonal to \( u_1, \ldots, u_{n-2} \) and to \( \nu_n \), so \( R[\nu_{n-1}] \geq \lambda_{n-1} \);

\vdots

a vector \( \nu_1 \) which is \( B \)-orthogonal to \( \nu_n, \ldots, \nu_2 \), and \( R[\nu_1] \geq \lambda_1 \).

In the sum: \( \lambda_1 + \cdots + \lambda_n \leq TR[L_n] \). In addition,

\[
TR[L(u_1, \ldots, u_n)] = \lambda_1 + \cdots + \lambda_n;
\]

(3) follows. Similarly,

\[
(4) \quad \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} = \max_{\text{choice of } L_n} TRinv[L_n].
\]

3. Recurrent characterization of \( \sum_{k+1}^{k+n} \lambda_i \) and \( \sum_{k+1}^{k+n} \lambda_i^{-1} \):

\[
(5) \quad \sum_{k+1}^{k+n} \lambda_i = \min_{\text{choice of } L_n \text{ } B \text{-orthogonal to } L(u_1, \ldots, u_k)} TR[L_n].
\]

In effect: in all \( tel \ L_n \) there exists a vector \( \nu_{k+n} \) \( B \)-orthogonal to \( u_1, \ldots, u_{k+n-1} \), therefore \( R[\nu_{k+n}] \geq \lambda_{k+n} \); etc.

\[
(6) \quad \sum_{k+1}^{k+n} \frac{1}{\lambda_i} = \max_{\text{choice of } L_n \text{ } A \text{-orthogonal to } L(u_1, \ldots, u_k)} TRinv[L_n].
\]

4. Direct characterizations of \( \sum_{k+1}^{k+n} \lambda_i \) and \( \sum_{k+1}^{k+n} \lambda_i^{-1} \)

4.1. Extremal property “in the style of Poincaré”.

\[
(7) \quad \sum_{k+1}^{k+n} \lambda_i = \min_{\text{choice of } L_{k+n} \subset L} \max_{\text{choice of } L_n} TR[L_n];
\]

\[
(8) \quad \sum_{k+1}^{k+n} \frac{1}{\lambda_i} = \max_{\text{choice of } L_{k+n} \subset L} \min_{\text{choice of } L_n} TRinv[L_n].
\]
4.2. Extremal property “in the style of Courant-Weyl”.

\[ \sum_{k+1}^{k+n} \lambda_i = \max \text{ choice of } L_k \text{ choice of } L_n \min \text{ B-orthogonal to } L_k TR[L_n]; \]

\[ \sum_{k+1}^{k+n} \frac{1}{\lambda_i} = \min \text{ choice of } L_k \text{ choice of } L_n \max \text{ A-orthogonal to } L_k TR^{inv}[L_n]. \]


5.1. Schrödinger-type equation. \( \nabla u + [\lambda - W(x, y, z)]u = 0 \) with certain fixed conditions on limits;

\( R^{(W)}[\nu] = \frac{D(\nu) + \iint W \nu^2 \,d\tau}{\iint \nu^2 \,d\tau}, \)

where \( d\tau \) is the volume element and \( D(\nu) \) is the Dirichlet integral.

\[ \sum_{i=1}^{n} \left( \lambda_{k_1+i}^{(W_1)} + \lambda_{k_2+i}^{(W_2)} - 2 \lambda_{k_1+k_2+i}^{(W_1+W_2)/2} \right) \leq 0 \quad (k_1 \geq 0, k_2 \geq 0, n \geq 1). \]

**Proof.** Denote by \( \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \ldots \) the eigenfunctions of \( \tilde{W}(x, y, z) = (W_1 + W_2)/2 \); in \( L(\tilde{u}_1, \ldots, \tilde{u}_{k_1+k_2+n}) \), there is an \( L_n \) orthogonal to both \( L(u_1^{(W_1)}, \ldots, u_{k_1}^{(W_1)}) \) and \( L(u_1^{(W_2)}, \ldots, u_{k_2}^{(W_2)}) \); thus, under the conditions of paragraph 3,

\[ \sum_{k_1+1}^{k_1+n} \lambda_{k_1+i}^{(W_1)} + \sum_{k_2+1}^{k_2+n} \lambda_{k_1+i}^{(W_2)} \leq TR^{(W)}[L_n] + TR^{(W_2)}[L_n] = 2TR^{(\tilde{W})}[L_n] \leq 2 \sum_{k_1+k_2+1}^{k_1+k_2+n} \lambda_{k_1+k_2+i}^{(W_1)}. \]

For \( k_1 = k_2 = 0 \), we have a convex inequality of the type of Pólya-Schiffer [1]; for \( n = 1 \), we have an inequality of the type of Weyl [2].

5.2. Inhomogeneous vibrating system. \( \mathcal{L}[u] - \lambda \rho(x, y, \ldots) = 0 \) with certain fixed conditions on the boundary, and with density \( \rho \geq 0 \). (Here \( \mathcal{L} \) is a self-adjoint linear differential operator). The Rayleigh quotient is

\[ R^{(\rho)}[\nu] = \int \nu \mathcal{L}[\nu] \,d\tau / \int \rho \nu^2 \,d\tau. \]

\[ \sum_{i=1}^{n} \left( \frac{1}{\lambda_{k_1+1}} + \cdots + \frac{1}{\lambda_{k_N+i}} - \frac{1}{\lambda_{k_1+\cdots+k_{N-1}+i}} \right) \geq 0. \]

For \( N = 2 \) and \( k_1 = k_2 = 0 \), this gives a convex inequality of the type of Pólya-Schiffer [1]; for \( n = 1 \), this gives an inequality of the type of Weyl [2]. If \( k_1 = k_2 = \ldots = k_N = 0 \) and \( n = \infty \), there is equality; we will return to this.
References


[3] We always suppose $B$ is positive definite; the relations (1), (3), (5), (7), (9), (11) remain valid if $A$ is indefinite with only finitely many negative eigenvalues.

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