Math 517: Day 29

Thm: Suppose $F: E \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then $F \in C^1(E, \mathbb{R}^m) \iff$ all $D_j f_i$ exist & are cts on E.

Pf: ($\Rightarrow$) Suppose $F \in C^1(E, \mathbb{R}^m)$. Let $\vec{x} \in E$ & let $\varepsilon > 0$. Then, by cty of F', there exists $\delta > 0$ s.t. $|\vec{x} - \vec{y}| < \delta$ implies

$$\| F'(\vec{x}) - F'(\vec{y}) \|_\infty < \varepsilon$$

Now, since F is diffble, we know $F'(\vec{x})\vec{e}_j = \sum_{i=1}^{m} (D_j F_i)(\vec{x})\, \vec{u}_i$ & , in particular, that all the $D_j F_i$ exist, so all that remains is to show they are cts.

Claim: If $|\vec{x} - \vec{y}| < \delta$, then $|D_j F_i(\vec{x}) - D_j F_i(\vec{y})| < \varepsilon$; in particular, $D_j F_i$ is cts $\forall\ i, j$.

The strategy is to show the RHS is less than $\| F'(\vec{x}) - F'(\vec{y}) \|$, which we know from above is $< \varepsilon$.

Now, $F'(\vec{x})\vec{e}_j = \sum_{i=1}^{m} (D_j F_i)(\vec{x})\, \vec{u}_i$ implies

$$(D_j F_i)(\vec{x}) = \left(F'(\vec{x})\vec{e}_j\right) \cdot \vec{u}_i$$

for all i, j. But then

$$|(D_j F_i)(\vec{x}) - (D_j F_i)(\vec{y})| = \left| \left(F'(\vec{x})\vec{e}_j\right) \cdot \vec{u}_i - \left(F'(\vec{y})\vec{e}_j\right) \cdot \vec{u}_i \right| = \left| \left[\left(F'(\vec{x}) - F'(\vec{y})\right)\vec{e}_j \right] \cdot \vec{u}_i \right| \le \left| \left(F'(\vec{x}) - F'(\vec{y})\right)\vec{e}_j \right| \cdot \underbrace{|\vec{u}_i|}_{\to 1}$$

$$\le \| F'(\vec{x}) - F'(\vec{y}) \| < \varepsilon.$$

($\Leftarrow$) For the other direction, it suffices to prove the theorem for $f: E \subseteq \mathbb{R}^n \to \mathbb{R}$ since we can build the matrix $F'(\vec{x})$ from the individual rows $F_i'(\vec{x})$.

So now, let $\vec{x} \in E$ & $\varepsilon > 0$. Since E open, $\exists$ a nbhd $N_r(\vec{x}) \subseteq E$. In fact, since $D_j f$ is cts for each j, $\exists\ r_j \le r$ s.t. $|\vec{y} - \vec{x}| < r_j \Rightarrow |D_j f(\vec{y}) - D_j f(\vec{x})| < \frac{\varepsilon}{n}$.

By taking $\min(r_1, \dots, r_n) = \tilde{r}$, we get a nbhd $S = N_{\tilde{r}}(\vec{x})$ s.t. $|D_j f(\vec{y}) - D_j f(\vec{x})| < \varepsilon/n$ $\forall\ \vec{y} \in S$ & all $j = 1, \dots, n$.

Now, for any $\vec{h} \in \mathbb{R}^n$ w/ $|\vec{h}| < \tilde{r}$, we know $\vec{x} + \vec{h} \in S$. In fact, since $\vec{h} = \sum_{j=1}^{n} h_j \vec{e}_j$, we can form the partial sums

$\vec{v}_0 = \vec{0}$ & $\vec{v}_k = h_1\vec{e}_1 + \dots + h_k\vec{e}_k$ for $k = 1, \dots, n$. Notice $\vec{v}_k = \vec{v}_{k-1} + h_k\vec{e}_k$.

So then $f(\vec{x} + \vec{h}) - f(\vec{x}) = \sum_{j=1}^{n} \left(f(\vec{x} + \vec{v}_j) - f(\vec{x} + \vec{v}_{j-1}) \right)$

Now the idea is that $f(\vec{x} + \vec{v}_j) - f(\vec{x} + \vec{v}_{j-1})$ is really a 1-dim'l fctn. The line segment $\vec{x} + \vec{v}_{j-1}$ to $\vec{x} + \vec{v}_j$ lies in S (since S is convex) & the restriction of f to that segment is cts & diffble (since $D_j f$ is cts.)

But then the Mean Value Thm $\Rightarrow$

$$f(\vec{x}+\vec{v}_j) - f(\vec{x}+\vec{v}_{j-1}) = |\vec{v}_j - \vec{v}_{j-1}| \, \frac{\partial}{\partial \vec{e}_j}\left(f(\vec{x}+\vec{v}_{j-1} + t_j h_j \vec{e}_j\right) \text{ for some } 0 \le t_j \le 1.$$

$$= h_j (D_j f)(\vec{x}+\vec{v}_{j-1} + t_j h_j \vec{e}_j)$$

Since $|\vec{x}+\vec{v}_{j-1} + t_j h_j \vec{e}_j - \vec{x}| = |\vec{v}_{j-1} + t_j h_j \vec{e}_j| \le |\vec{v}_j| < \tilde{r}$, we have

$$|(D_j f)(\vec{x}+\vec{v}_{j-1} + t_j h_j \vec{e}_j) - (D_j f)(\vec{x})| < \frac{\varepsilon}{n}$$

So we see that

$$|f(\vec{x}+\vec{v}_j) - f(\vec{x}+\vec{v}_{j-1}) - h_j (D_j f)(\vec{x})| = |h_j (D_j f)(\vec{x}+\vec{v}_{j-1} + t_j h_j \vec{e}_j) - h_j (D_j f)(\vec{x})| < |h_j| \frac{\varepsilon}{n}$$

& hence

$$|f(\vec{x}+\vec{h}) - f(\vec{x}) - \textstyle\sum h_j (D_j f)(\vec{x})| = |\sum (f(\vec{x}+\vec{v}_j) - f(\vec{x}+\vec{v}_{j-1})) - \sum h_j (D_j f)(\vec{x})| \le \sum |f(\vec{x}+\vec{v}_j) - f(\vec{x}+\vec{v}_{j-1}) - h_j (D_j f)(\vec{x})| < \frac{1}{n} \sum |h_j| \varepsilon \le |\vec{h}| \varepsilon$$

for all $\vec{h}$ w/ $|\vec{h}| < \tilde{r}$.

But this says f is diffble at $\vec{x}$ & $f'(\vec{x})\vec{h} = \sum h_j (D_j f)(\vec{x})$, so the matrix of $f'(\vec{x})$ is

$$\begin{bmatrix} (D_1 f)(\vec{x}) & (D_2 f)(\vec{x}) & \cdots & (D_n f)(\vec{x}) \end{bmatrix}$$

which is clearly cts as a fctn of $\vec{x}$ since the $D_j f$ are.