Math 517 HW #7

Due 12:00 PM Friday, Oct. 20

1. (Rudin Problem 7.3) Give an example of sequences $\{f_n\}, \{g_n\}$ of uniformly converging functions on some set E so that $\{f_n g_n\}$ does not converge uniformly on E.

2. (Rudin Problem 7.7) Define $f_n(x) = \frac{x}{1+nx^2}$ for each $n = 1, 2, \ldots$.

   (a) Show that $\{f_n\}$ converges uniformly to a function f on $\mathbb{R}$.

   (b) Show that $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \neq 0$, but that this fails when $x = 0$.

3. Prove that every uniformly convergent sequence of bounded real-valued functions is uniformly bounded (i.e., there exists $M > 0$ so that $|f_n(x)| < M$ for all n and all x.)

4. A family $\mathcal{F}$ of (real- or complex-valued) functions on a set E in a metric space X is *equicontinuous* on E if for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - f(y)| < \epsilon$$

   whenever $d(x, y) < \delta$, $x, y \in E$, and $f \in \mathcal{F}$.

   Give an example of an equicontinuous sequence $\{f_n\}$ of functions on some metric space that converges pointwise but not uniformly.