Math 517 HW #4

Due 12:00 PM Friday, Sept. 22

1. (Rudin Problem 3.10) Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are nonzero. Prove that the radius of convergence of the power series is at most 1.

2. (Rudin Problem 3.23) Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. (Note that this new sequence lives in $\mathbb{R}$.)

3. (Rudin Problem 3.24) Let X be a metric space.

   (a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if
   $$\lim_{n\to\infty} d(p_n, q_n) = 0.$$
   Prove that this is an equivalence relation.

   (b) Let X^* be the set of all equivalence classes from (a). If $P \in X^*$ has representative $\{p_n\}$ and $Q \in X^*$ has representative $\{q_n\}$, define
   $$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n),$$
   which exists by the previous exercise. Show that $\Delta(P, Q)$ is independent of the choice of representatives for the equivalence classes P and Q, so it is a well-defined distance on X^*.

   (c) Prove that the metric space (X^*, Δ) is complete (i.e., every Cauchy sequence in X^* converges).

   (d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the equivalence class of this sequence. Prove that
   $$\Delta(P_p, P_q) = d(p, q)$$
   for each $p, q \in X$. In other words, the map $\phi : X \to X^*$ is an *isometry* onto its image (an isometry is a distance-preserving map).

   (e) Prove that $\phi(X)$ is dense in X^*, and that $\phi(X) = X^*$ if X is complete. From (d), we can identify X and $\phi(X)$, and so we see X as densely embedded in the complete metric space X^*. X^* is called the *completion* of X.

   [*Remark:* To formally construct $\mathbb{R}$, we can simply define $\mathbb{R} = \mathbb{Q}^*$, the completion of the metric space $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|$. As mentioned in class, completing $\mathbb{Q}$ with respect to the p-adic metric d_p produces the field $\mathbb{Q}_p$ of p-adic numbers.]

4. (Rudin Problem 4.16) For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the floor function; i.e., $\lfloor x \rfloor$ is the largest integer less than or equal to x. Let $(x) := x - \lfloor x \rfloor$ be the fractional part of x. For which x is $\lfloor x \rfloor$ continuous? For which x is (x) continuous?