Math 517 HW #10

Due 12:00 PM Friday, Nov. 10

1. (Rudin Problem 9.9) If $E \subseteq \mathbb{R}^n$ is a connected open set and $F : E \to \mathbb{R}^m$ is differentiable such that $F'(\vec{x}) = \vec{0}$ for all $\vec{x} \in E$, prove that F is constant on E.

2. (Rudin Problem 9.12) Fixe two real numbers $0 < a < b$. Define $F : \mathbb{R}^2 \to \mathbb{R}^3$ where $F(s,t) = (f_1(s,t), f_2(s,t), f_3(s,t))$ with

$$\begin{aligned} f_1(s,t) &= (b + a\cos s)\cos t \\ f_2(s,t) &= (b + a\cos s)\sin t \\ f_3(s,t) &= a\sin s. \end{aligned}$$

   (a) Describe the range T of F (it is a compact subset of $\mathbb{R}^3$).

   (b) Show that there are exactly 4 points $\vec{p} \in T$ such that

$$(\nabla f_1)(F^{-1}(\vec{p})) = \vec{0}.$$

   (c) Determine the set of all $\vec{q} \in T$ such that

$$(\nabla f_3)(F^{-1}(\vec{q})) = \vec{0}.$$

   (d) Show that one of the points $\vec{p}$ found in part (b) corresponds to a local maximum of f_1, one corresponds to a local minimum, and that the other two are neither (they are saddle points).

   Which of the points $\vec{q}$ found in part (c) correspond to maxima or minima?

   (e) Let $\lambda \in \mathbb{R}$ be irrational, and define $G(t) = F(t, \lambda t)$. Prove that G is an injective mapping of $\mathbb{R}$ onto a dense subset of T, and show that

$$|G'(t)|^2 = a^2 + \lambda^2(b + a\cos t)^2.$$

3. Let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f_1(x,y) = e^x \cos y, \quad f_2(x,y) = e^x \sin y.$$

   (a) What is the range of F?

   (b) Show that the Jacobian of F is not zero at any point of $\mathbb{R}^2$, so every point of $\mathbb{R}^2$ has a neighborhood on which F is injective. However, F is not injective globally.

   (c) Put $\vec{a} = (0, \pi/3)$, $\vec{b} = F(\vec{a})$, and let G be the continuous inverse of F defined in a neighborhood of $\vec{b}$ so that $G(\vec{b}) = \vec{a}$. Find an explicit formula for G, compute $F'(\vec{a})$ and $G'(\vec{b})$, and verify that they satisfy the equation

$$G'(\vec{b}) = \left[F'(G(\vec{b}))\right]^{-1}$$

   that came up in the proof of the Inverse Function Theorem.

   (d) What are the images under F of lines parallel to the coordinate axes of $\mathbb{R}^2$?