Math 474: Dg 7

Notice that, from the previous discussion:

Proposition: Any 2 curves w/ the same tangent indicatrix (up to reparametrization) have the same total curvature $\int \kappa ds$ & same $\int \sqrt{\kappa^2+\tau^2}\, ds$

Now we can relate global features of the original curve to the global geometry of the tangent indicatrix. For example:

Thm: A spherical curve γ is the tangent indicatrix of a closed curve $\alpha \Leftrightarrow \gamma$ crosses **every** plane through the center of the sphere.

Proof: ($\Rightarrow$) Suppose $\gamma = T(s)$ for a closed curve $\alpha(s)$. If α has length L, then

$$\vec{0} = \alpha(L) - \alpha(0) = \int_0^L \alpha'(s)\, ds = \int_0^L T(s)\, ds$$

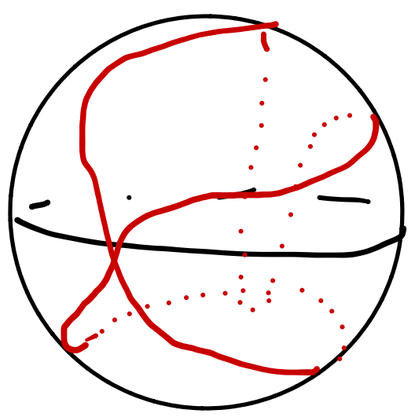
so $\gamma(s) = T(s)$ has center of mass at $\vec{0}$. Moreover, if P is a plane through $\vec{0}$ w/ normal vector $\vec{n}$, then

$$0 = \langle \vec{n}, \int_0^L T(s) ds \rangle = \int_0^L \langle \vec{n}, T(s) \rangle ds = \int_0^L \langle \vec{n}, \gamma(s) \rangle ds$$

so (by the Mean Value Theorem for integrals) $\exists\, s^*$ s.t. $\langle \vec{n}, \gamma(s^*) \rangle = 0$.

But then $\gamma(s^*) \perp \vec{n}$, so $\gamma(s^*) \in P$.

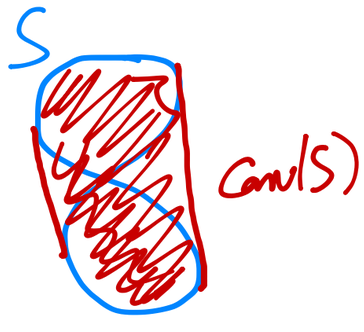
not $T(s)$ for a closed curve α



$T(s)$ for some closed α

($\Leftarrow$) This is the harder direction. Now we're assuming $\gamma(s)$ cuts any plane through $\vec{0}$.

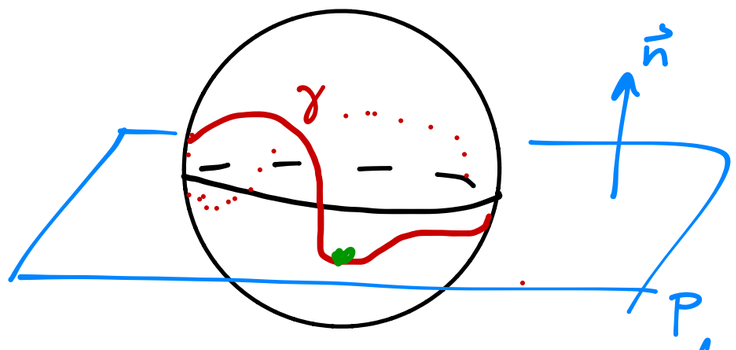
Def: the **convex hull** of a set $S \subseteq \mathbb{R}^n$ is the intersection of all halfspaces containing S. The big idea is:



It's a (very!) good theorem that $\text{Conv}(S) = \{p : p = \int_S d\mu\}$ where $d\mu \geq 0$ on S. In particular, for a curve γ,

$$\text{Conv}(\gamma) = \{p : p = \int w(s)\gamma(s)\, ds\}$$

where $w(s) \geq 0$ is a weight fctn & s is arclength of γ.



Now the goal is to show $\vec{0} \in \text{conv}(\gamma)$. Let h be a halfspace containing γ w/ supporting plane P w/ normal $\vec{n}$.

Translate in the $\vec{n}$ direction until the plane contacts γ. All subsequent planes will intersect γ until we have contact forever at the top of γ.

Since the plane with $\vec{0}$ w/ normal $\vec{n}$ does at γ, we see that, if a halfspace contains γ, it contains $\vec{0}$.

So $\vec{0} \in \text{conv}(\gamma)$, which implies $\exists\ w(s) \geq 0$ s.t. $\int w(s)\,\gamma(s)\,ds = \vec{0}$.

Thus if $dt = w(s)\,ds$, we can define the curve $\alpha(t) = \int_0^t \gamma(t)\,dt$ & see that

$$\alpha'(t) = \gamma(t) \quad \& \quad \int_0^L \alpha'(t)\,dt = \vec{0}.$$

So α is a closed curve w/ γ as its tangent indicatrix. ▨