Math 474: Day 39

We're in the midst of trying to work out the geodesics on the sphere, parametrized by $\vec{x}(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$

If we assume $\alpha(t) = \vec{x}(u(t), v(t))$ is a geodesic, then the geodesic equations tell us

$$v'(t) = \frac{C}{\cos^2 u} = C \sec^2 u \quad \text{and} \quad u' = \pm\sqrt{D^2 - \frac{C^2}{\cos^2 u}}$$

where C & D are constants.

Now, using the chain rule, $\frac{dv}{du} = \frac{v'(t)}{u'(t)} = \pm \frac{C \sec^2 u}{\sqrt{D^2 - \frac{C^2}{\cos^2 u}}} = \pm \frac{C \sec^2 u}{\sqrt{D^2 - C^2 \sec^2 u}}$

Recall $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \tan^2\theta + 1 = \sec^2\theta$, so we can rewrite the above as

$$dv = \pm \frac{C \sec^2 u \, du}{\sqrt{D^2 - C^2(1+\tan^2 u)}} = \pm \frac{C \sec^2 u \, du}{\sqrt{(D^2 - C^2) - C^2 \tan^2 u}}$$

Now we do something very sneaky: make the change of variables

$$C \tan u = \sqrt{D^2 - C^2} \cos w \Rightarrow C \sec^2 u \, du = -\sqrt{D^2 - C^2} \sin w \, dw$$

then $\sqrt{(D^2 - C^2) - C^2 \tan^2 u} = \sqrt{(D^2 - C^2) - (D^2 - C^2)\cos^2 w} = \sqrt{D^2 - C^2}\sqrt{1 - \cos^2 w} = \sqrt{D^2 - C^2} \sin w$

and we see that

$$dv = \pm \frac{-\sqrt{D^2 - C^2} \sin w \, dw}{\sqrt{D^2 - C^2} \sin w} = \mp dw$$

& therefore $w = \pm v + a$ for some constant a. Therefore,

$$C \tan u = \sqrt{D^2 - C^2} \cos w = \sqrt{D^2 - C^2} \cos(\pm v + a) = \sqrt{D^2 - C^2}\left(\cos(\pm v)\cos a - \sin(\pm v)\sin a\right) = \sqrt{D^2 - C^2}\left(\cos v \cos a \pm \sin v \sin a\right)$$

& so $C \sin u = \sqrt{D^2 - C^2} \cos u (A \cos v + B \sin v)$.

So now we try to remember what u & v even are; recall $\vec{x}(u,v) = (\cos u \cos v, \cos u \sin v, \sin u) = (X, Y, Z)$,

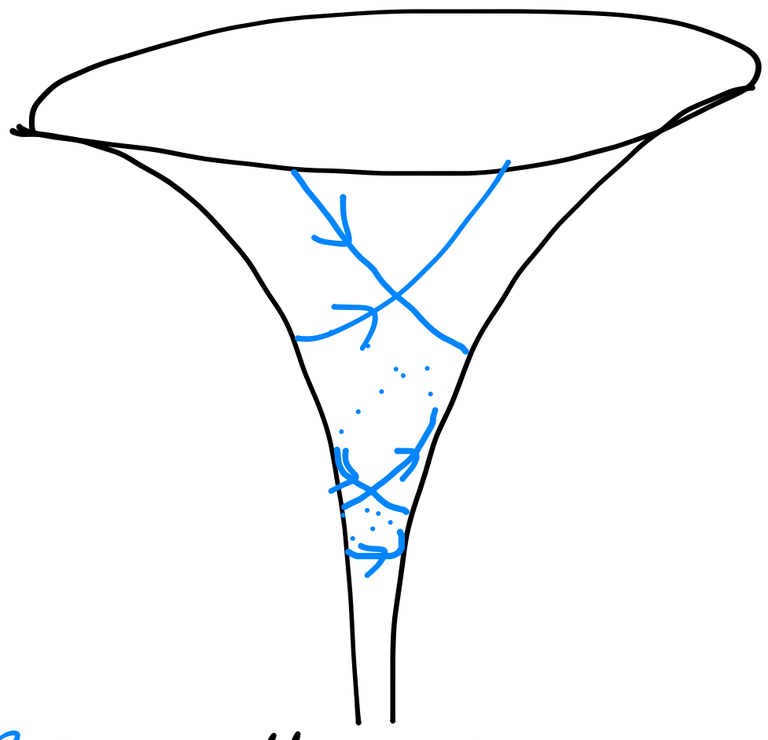
So our equation is saying $CZ = A\sqrt{D^2 - C^2} X + B\sqrt{D^2 - C^2} Y$, or

$$A\sqrt{D^2 - C^2} X + B\sqrt{D^2 - C^2} Y - CZ = 0,$$

which describes a plane through the origin! So the geodesics are exactly the great circles!

Clairaut's Relation: The geodesics on a surface of revolution satisfy $r\cos\varphi = \text{const.}$ where r is the distance to the axis & φ is the angle w/ a parallel. Moreover, any constant speed curve satisfying $r\cos\varphi = \text{const}$ which is not a parallel is a geodesic.

Corollary: A geodesic on a surface of revolution asymptotic to the axis reaches $z=-\infty \Leftrightarrow$ it is a meridian



Proof: We know $r\cos\varphi = C$ for some constant C. Now, $r\cos\varphi \le r$, so if $r \to 0$ as $z \to -\infty$, we eventually reach $r = C$.

At this point $\cos\varphi = 1$ so α is tangent to the parallel at that point & has to turn around. $\blacksquare$

Proof of Clairaut's Relation: We've seen that on a surface of revolution, $E=1$, $F=0$, $G=f(u)^2$ where

$$\vec{x}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $f'(u)^2 + g'(u)^2 = 1$. It's a nice computational exercise to show all Christoffel symbols are 0 except $\Gamma^v_{uv} = \frac{f'(u)}{f(u)}$ & $\Gamma^u_{vv} = -f(u)f'(u)$.

Then the geodesic eqns become

$$u'' - f(u)f'(u)(v')^2 = 0$$
$$v'' + 2\frac{f'(u)u'}{f}v' = 0$$

As w/ the sphere, we separate variables & integrate in the 2nd eqn:

$$\frac{v''}{v'} = -2\frac{f'(u)u'(t)}{f(u)}$$

so integrating wrt t gives

$$\ln(v'(t)) = -2\ln(f(u(t))) + C_1 = -2(\ln(f(u(t))) + \ln(e^{-C_1/2})) = -2\ln(e^{-C_1/2}\ln(f(u(t)))) = \ln\left(\frac{e^{C_1}}{f(u(t))}\right)$$

or

$$v'(t) = \frac{C}{f(u(t))^2} \quad \text{for } C = e^{C_1}.$$

Therefore, $C = f(u(t))^2 v'(t) = G v'(t)$

But then $\alpha'(t) = u'(t)\vec{x}_u + v'(t)\vec{x}_v$, so $\langle \alpha'(t), \vec{x}_v \rangle = v'(t)\langle \vec{x}_v, \vec{x}_v \rangle = v'(t) G = C$,

so $C = \langle \alpha', \vec{x}_v \rangle = |\alpha'(t)||\vec{x}_v| \cos\varphi = |\alpha'(t)|\,|f(u)| \cos\varphi = |\alpha'(t)|\, r \cos\varphi$.

But $|\alpha'(t)| = \text{const}$. Since α is a geodesic, so $r\cos\varphi = \text{const}$.

We'll finish off the proof next time...