Math 474 HW #4

Due 2:00 PM Friday, Sept. 29

1. (Shifrin Problem 1.3.9) Let $\alpha : [0, L] \to \mathbb{R}^3$ be parametrized by arclength. A *unit normal vector field* X is a vector-valued function with $X(0) = X(L)$, $X(s) \cdot T(s) = 0$, and $\|X(s)\| = 1$ for all s. Define the *twist* of X to be

$$\mathrm{tw}(\alpha, X) = \frac{1}{2\pi} \int_0^L X'(s) \cdot (T(s) \times X(s))\, \mathrm{ds}\,.$$

   (a) Show that if X and Y are two unit normal fields on α, then $\mathrm{tw}(\alpha, X)$ and $tw(\alpha, Y)$ differ by an integer.

   (b) From part (a), the fractional part of $\mathrm{tw}(\alpha, X)$ is independent of X, so depends only on α. We call this the *total twist* of α. Prove that the total twist of α equals the fractional part of $\frac{1}{2\pi} \int_0^L \tau\, \mathrm{ds}$.

   (c) Prove that if a closed curve lies on a sphere, then its total twist is 0.

2. For a *piecewise-smooth* closed curve in $\mathbb{R}^3$, we can define the total curvature of the curve to be

$$\int \kappa(s) ds + \sum_{i=1}^{n} \theta_i,$$

   where the θ_i are the exterior angles at the corners of the curve (see Shifrin's problem 1.3.12 for a precise definition of the exterior angle and a picture).

   Generalize Fenchel's Theorem to show that the total curvature of a piecewise-smooth curve is $\geq 2\pi$, with equality if and only if the curve is planar and convex.

3. (Milnor's integral geometric total curvature formula) Suppose $\alpha(s)$ is a curve in $\mathbb{R}^3$ with curvature $\kappa(s)$. For any $\vec{p} \in S^2$ (where S^2 denotes the unit sphere in $\mathbb{R}^3$), let $\alpha_{\vec{p}}(s)$ be the projection of α to the line determined by $\vec{p}$. This is not a regular curve, but we can define its total curvature as

$$\kappa_{\vec{p}} := \pi \cdot (\text{total number of times } \alpha_{\vec{p}} \text{ changes direction}).$$

   Prove that

$$\int \kappa(s) ds = \frac{1}{4\pi} \int_{\vec{p} \in S^2} \kappa_{\vec{p}}\, d\vec{p}.$$