Math 474 HW #2

Due 2:00 PM Friday, Sept. 8

1. (Shifrin Problem 1.2.3(a)–(d), (g)) Compute (T, κ, N, B, τ) for each of the following curves (please use a computer to help you):

   (a) $\alpha(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right)$, $s \in (-1, 1)$

   (b) $\alpha(t) = \left(\frac{1}{2}e^t(\sin t + \cos t), \frac{1}{2}e^t(\sin t - \cos t), e^t\right)$

   (c) $\alpha(t) = \left(\sqrt{1+t^2}, t, \ln(t + \sqrt{1+t^2})\right)$

   (d) $\alpha(t) = (e^t \cos t, e^t \sin t, e^t)$

   (e) $\alpha(t) = (t - \sin t \cos t, \sin^2 t, \cos t)$, $t \in (0, \pi)$

2. (Shifrin Problem 1.2.8) Let α be a regular curve parametrized by arclength with nonvanishing curvature. The *normal line* to α at $\alpha(s)$ is the line through $\alpha(s)$ with direction vector $N(s)$ (the Frenet normal). Suppose all normal lines to α pass through a given fixed point. What can you say about the curve α? Does your answer change if the curve is not regular?

3. (Shifrin Problem 1.2.11) Suppose $\alpha(t)$ is a regular curve, not necessarily parametrized by arclength. Show that the torsion of α is given by

$$\tau = \frac{\alpha' \cdot (\alpha'' \times \alpha''')}{\|\alpha' \times \alpha''\|^2}.$$

4. (Shifrin Problem 1.2.20) Two distinct parametrized curves α and β are called *Bertrand mates* if, for each t, the normal line to α at $\alpha(t)$ is the same as the normal line to β at $\beta(t)$.

   Suppose α and β are Bertrand mates.

   (a) If α is parametrized by arclength, show that $\beta(s) = \alpha(s) + rN(s)$ for some constant r, meaning that corresponding points on α and β are a constant distance apart.

   (b) Show that, moreover, the angle between the tangent vectors to α and β at corresponding points is constant. (Hint: consider the dot product)

   (c) Suppose α is parametrized by arclength and that both κ and τ are nonvanishing. Show that α has a Bertrand mate β if and only if there are constants r and c so that $r\kappa + c\tau = 1$.

   (d) Show that a curve α that has more than one Bertrand mate must be a helix (and hence have infinitely many Bertrand mates).