Math 474 Final Exam

Due December 18, 2015 at 4:00 PM.

Instructions: This take-home exam covers the material covered in class this semester. **You may use:** Maple/Mathematica/MATLAB, a calculator, Shifrin, your notes, notes posted on the website, your brain. **You may not use:** The internet, other books, other people's notes, other people's brains.

1. Suppose $\alpha(s)$ is a closed, regular curve parametrized by arc length which lies on the unit sphere S^2 and has length ℓ. In other words, the closure and length conditions mean that $\alpha(0) = \alpha(\ell)$, and the fact that it lies on the sphere means that $|\alpha(s)| = 1$ for all $s \in [0, \ell]$.

   Assume the curvature $\kappa(s)$ is never zero and prove that

   $$\int_0^\ell \tau(s)\,\mathrm{d}s = 0.$$

   In other words, the total torsion of any closed, arclength parametrized spherical curve is zero. (Hint: consider the angle $\theta(s)$ between the Frenet frame normal $\vec{\mathbf{n}}(s)$ and the surface normal $\vec{\mathbf{N}}(s)$ at the point $\alpha(s)$ on the curve. It will be helpful to think about how $\theta(s)$ relates to the angle between $\vec{\mathbf{N}}(s)$ and the Frenet frame binormal $\vec{\mathbf{b}}(s)$.)

2. The Mercator projection is a map of the Earth that (somewhat infamously) makes regions near the poles look disproportionally larger than regions near the equator. One way to think of a map projection is as a system of local coordinates on the sphere. In the case of the Mercator projection, the local coordinate map is

   $$\vec{\mathbf{x}}(u, v) = (\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, \tanh v),$$

   where sech and tanh are the hyperbolic secant and tangent, respectively, and $u \in (-\pi, \pi)$, $v \in \mathbb{R}$.

   Since the Mercator projection distorts areas, it is definitely not an isometry, but the reason it has persisted for hundreds of years is that it is *conformal*, meaning angles measured in the uv-plane (i.e., on the map) agree with angles measured on the sphere. Prove that the Mercator projection is conformal.

   (The following hyperbolic trig identities may come in handy: $\operatorname{sech} x = \frac{1}{\cosh x}$, $\tanh x = \frac{\sinh x}{\cosh x}$, $\frac{d}{dx} \sinh x = \cosh x$, $\frac{d}{dx} \cosh x = \sinh x$, $\cosh^2 x - \sinh^2 x = 1$.)

3. Suppose α is a line of curvature on a regular surface Σ and that α is planar. Is it true that α must be a straight line? Either prove that it is, or give an explicit example of a surface Σ and a line of curvature α which is planar but not a straight line.

4. Consider the paraboloid Σ given by $\vec{\mathbf{x}}(u, v) = (u \cos v, u \sin v, u^2)$ and let Σ_r be the subsurface with $0 \le u \le r$.

   (a) Compute the geodesic curvature of the boundary $\partial\Sigma_r$ of Σ_r and use this to determine the total geodesic curvature $\int_{\partial\Sigma_r} \kappa_g(s)\,\mathrm{d}s$ of the boundary.

   (b) Determine the Euler characteristic $\chi(\Sigma_r)$.

   (c) Use the Gauss–Bonnet theorem to compute $\iint_{\Sigma_r} K d\text{Area}$.

   (d) Compute $\iint_{\Sigma_r} k d\text{Area}$ explicitly by calculating the Gaussian curvature of Σ_r (be careful in the integral to integrate with respect to dArea and not with respect to $du\,dv$.)