Math 369 Exam #3 Practice Problem Solutions

1. Diagonalize the matrix
$$A = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}.$$
In other words, find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$.
Answer: We need to compute the eigenvalues and eigenvectors of A. For a 2×2 matrix like this, we can easily just determine the characteristic polynomial and find its roots. The characteristic polynomial is
$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -2 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) + 2 = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4).$$
Therefore, the eigenvalues are 3 and 4, and now we can find eigenvectors.
The eigenvectors for $\lambda = 3$ are solutions to
$$(A - 3I)\vec{v} = \vec{0},$$
so we solve the augmented matrix
$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right).$$
But now subtracting twice row 1 from row 2 gives
$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$
so solutions are the vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $-x_1 + x_2 = 0$, meaning that $x_1 = x_2$. For example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is such an eigenvector.
On the other hand, the eigenvectors for $\lambda = 4$ are solutions to
$$(A - 4I)\vec{v} = \vec{0},$$
so we solve the augmented matrix
$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right].$$
But now subtracting row 1 from row 2 yields
$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$
so solutions are the vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $-2x_1 + x_2 = 0$, meaning that $x_1 = 2x_2$. For example, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is such an eigenvector.

Therefore, the eigenvectors form the columns of the matrix

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and D is the diagonal matrix whose nonzero entries are the eigenvalues

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

2. Let A be an $n \times n$ matrix.

   (a) Let $\lambda \in \mathbb{R}$ be a number. Give the definition of the eigenspace V_λ of A associated to λ.
   Answer: The eigenspace is defined to be

   $$V_\lambda = \{\vec{v} \in V : A\vec{v} = \lambda\vec{v}\}.$$

   (b) Show that $\vec{0} \in V_\lambda$ (this is true regardles of what λ is).

   Proof. Regardless of what A is, $A\vec{0} = \vec{0} = \lambda\vec{0}$, so $\vec{0} \in V_\lambda$. □

   (c) Suppose that $\lambda \neq \tau$, but that $\vec{v} \in V_\lambda$ and $\vec{v} \in V_\tau$. Prove that this means $\vec{v} = \vec{0}$.

   Proof. Since $\vec{v} \in V_\lambda$ and $\vec{v} \in V_\tau$, we know that

   $$A\vec{v} = \lambda\vec{v} \quad \text{and} \quad A\vec{v} = \tau\vec{v}.$$

   But since $A\vec{v} = A\vec{v}$, this means that

   $$\lambda\vec{v} = \tau\vec{v}.$$

   In particular, if v_i is the ith entry in $\vec{v}$, this tells us that $\lambda v_i = \tau v_i$; since $\lambda \neq \tau$, this can only happen if $v_i = 0$. Since the same argument works for all i, this implies that $\vec{v} = \vec{0}$. □

3. Let V be a vector space with an inner product. Let v be some particular element of V and define

   $$W = \{w \in V : \langle w, v\rangle = 0\}.$$

   Prove that W is a subspace of V.

   Proof. As usual, we need to show that W is closed under addition and scalar multiplication.

   First, suppose $w_1, w_2 \in W$. The goal is to show that $w_1 + w_2 \in W$. Since $w_1, w_2 \in W$, we know that

   $$\langle w_1, v\rangle = 0 \quad \text{and} \quad \langle w_2, v\rangle = 0.$$

   Therefore, using the additivity of the inner product,

   $$\langle w_1 + w_2, v\rangle = \langle w_1, v\rangle + \langle w_2, v\rangle = 0 + 0 = 0,$$

so $w_1 + w_2 \in W$.

Also, if $w \in W$ and $\lambda \in \mathbb{R}$, then we know that $\langle w, v\rangle = 0$, so it follows from homogeneity of the inner product that

$$\langle \lambda w, v\rangle = \lambda\langle w, v\rangle = \lambda \cdot 0 = 0,$$

so $\lambda w \in W$.

Therefore, W is closed under addition and scalar multiplication, so it is a subspace of V. $\square$

4. Consider $\mathbb{R}^3$ with the slightly unusual inner product

$$\langle \vec{u}, \vec{v}\rangle = \frac{1}{6}u_1v_1 + \frac{1}{8}u_2v_2 + \frac{1}{27}u_3v_3.$$

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

(a) Show that $\{\vec{v}_1, \vec{v}_2\}$ is an orthonormal set with respect to this weird inner product.

Proof. We just need to compute $\langle \vec{v}_1, \vec{v}_1\rangle$, $\langle \vec{v}_2, \vec{v}_2\rangle$, and $\langle \vec{v}_1, \vec{v}_2\rangle$:

$$\begin{aligned}
\langle \vec{v}_1, \vec{v}_1\rangle &= \frac{1}{6}\cdot 1\cdot 1 + \frac{1}{8}\cdot 2\cdot 2 + \frac{1}{27}\cdot 3\cdot 3 = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1 \\
\langle \vec{v}_2, \vec{v}_2\rangle &= \frac{1}{6}\cdot 1\cdot 1 + \frac{1}{8}\cdot (-2)\cdot (-2) + \frac{1}{27}\cdot 3\cdot 3 = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1 \\
\langle \vec{v}_1, \vec{v}_2\rangle &= \frac{1}{6}\cdot 1\cdot 1 + \frac{1}{8}\cdot 2\cdot (-2) + \frac{1}{27}\cdot 3\cdot 3 = \frac{1}{6} - \frac{1}{2} + \frac{1}{3} = 0,
\end{aligned}$$

so $\vec{v}_1$ and $\vec{v}_2$ have norm 1 and are orthogonal with respect to this weird inner product. $\square$

(b) Let $\vec{u}_3 = \begin{bmatrix} 0 \\ 8 \\ 18 \end{bmatrix}$. Apply the Gram-Schmidt procedure to the set $\{\vec{v}_1, \vec{v}_2, \vec{u}_3\}$ to get an orthonormal set. (*Hint:* you already know from part (a) that $\{\vec{v}_1, \vec{v}_2\}$ is orthonormal, so $\vec{u}_3$ is the only vector that needs adjustment.)

Answer: Since we already have $\vec{v}_1$ and $\vec{v}_2$, we only need to figure out $\vec{v}_3$. Recall that we find $\vec{v}_3$ in a two-step process: first, we compute

$$\begin{aligned}
\vec{w}_3 &= \vec{u}_3 - \langle \vec{u}_3, \vec{v}_1\rangle\vec{v}_1 - \langle \vec{u}_3, \vec{v}_2\rangle\vec{v}_2 \\
&= \begin{bmatrix} 0 \\ 8 \\ 18 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 8 \\ 18 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \\
&= \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix}.
\end{aligned}$$

Then, we let

$$\vec{v}_3 = \frac{1}{\|\vec{w}_3\|}\vec{w}_3 = \frac{1}{\sqrt{4}}\begin{bmatrix}-4\\0\\6\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-4\\0\\6\end{bmatrix} = \begin{bmatrix}-2\\0\\3\end{bmatrix}$$

Therefore, the desired orthonormal set is

$$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}-2\\0\\3\end{bmatrix}\right\}.$$

5. Solve the system of differential equations

$$\begin{aligned} y_1' &= -2y_1 + 2y_2 \\ y_2' &= y_1 - y_2. \end{aligned}$$

Answer: As usual, the strategy is to diagonalize the matrix $A = \begin{bmatrix}-2 & 2\\1 & -1\end{bmatrix}$ as $A = PDP^{-1}$, solve the simpler system $\vec{u}' = D\vec{u}$, and then to recover the solution of the original system as $\vec{y} = P\vec{u}$.

So first we compute eigenvalues of A, which are the roots of the characteristic polynomial

$$\det(A-\lambda I) = \begin{vmatrix}-2-\lambda & 2\\1 & -1-\lambda\end{vmatrix} = (-2-\lambda)(-1-\lambda)-2 = \lambda^2+3\lambda+2-2 = \lambda^2+3\lambda = \lambda(\lambda+3),$$

so the eigenvalues are $\lambda = -3$ and $\lambda = 0$.

Now we look for eigenvectors.

$\lambda = -3$**:** We need to solve $(A+3I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c}1 & 2 & 0\\1 & 2 & 0\end{array}\right].$$

It's clear that row-reducing yields the equation $x_1 + 2x_2 = 0$, so $x_1 = -2x_2$, and the corresponding eigenvector is $\begin{bmatrix}-2\\1\end{bmatrix}$.

$\lambda = 0$**:** We're solving $(A-0I)\vec{x} = \vec{0}$, or simply $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c}-2 & 2 & 0\\1 & -1 & 0\end{array}\right].$$

Adding half of row 1 to row 2 yields

$$\left[\begin{array}{cc|c}-2 & 2 & 0\\0 & 0 & 0\end{array}\right],$$

so our equation is $-2x_1 + 2x_2 = 0$, or just $x_1 = x_2$, and the eigenvector we're looking for is $\begin{bmatrix}1\\1\end{bmatrix}$.

Therefore,

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, our simplified system is just

$$\begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} -3u_1(x) \\ 0 \end{bmatrix}.$$

Therefore, $u_1(x) = C_1 e^{-3x}$ and $u_2(x) = C_2$. Hence, the solutions to our original system are

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{-3x} \\ C_2 \end{bmatrix} = \begin{bmatrix} -2C_1 e^{-3x} + C_2 \\ C_1 e^{-3x} + C_2 \end{bmatrix}.$$

6. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

   (a) Every invertible matrix can be diagonalized.
   Answer: False. Consider the matrix

   $$A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}.$$

   Then $\det A = 1$, so A is indeed invertible. Now, if we try to find the eigenvalues of A, we solve

   $$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 5 \\ 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)(1-\lambda), \end{aligned}$$

   so $\lambda = 1$. Since A has only one eigenvalue, there must be two linearly independent eigenvectors associated with the eigenvalue 1 – otherwise there won't be enough columns to make an invertible matrix P.
   The eigenvector(s) corresponding to 1 will be in the nullspace of

   $$A - I = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$$

   which is already in reduced echelon form. If $(A - I)\vec{x} = 0$, then it must be the case that $x_2 = 0$, so eigenvectors are of the form

   $$\begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

   Therefore, there are not 2 linearly independent eigenvectors, so A is not diagonalizable.

   (b) Every diagonalizable matrix is invertible.
   Answer: False. The matrix

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

   is certainly diagonalizable (it is itself a diagonal matrix), but A is not invertible since it is of rank 1 (alternatively: $\det A = 0$).

(c) If the matrix A is not invertible, then 0 is an eigenvalue of A.

Answer: True. If A is not invertible, then rank $A < n$. Therefore, since

$$\dim \operatorname{col} A + \dim \operatorname{nul} A = n$$

and $\dim \operatorname{col} A = \operatorname{rank} A$, we see that the nullspace of A has dimension ≥ 1, so there is at least one non-zero vector $\vec{v}$ such that $A\vec{v} = \vec{0}$. Thus, 0 is an eigenvalue of A with corresponding eigenvector $\vec{v}$.

7. Let

$$B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 2 & 4 & 8 \end{bmatrix}$$

(a) Find an orthonormal basis (using the usual dot product) for the column space of B.

Answer: First of all, we need to determine some basis for $\operatorname{col}(B)$, so we want to row-reduce to determine the pivot columns of B. Subtracting twice row 1 from row 2 and from row 3 yields

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

Now we can subtract twice row 2 from row 3 to get

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the first and second columns of B are the pivot columns, so $\{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$ is a basis for $\operatorname{col}(B)$. To find an orthonormal basis, we do the Gram-Schmidt process. First, let

$$\vec{v}_1 = \frac{1}{\|\vec{u}_1\|}\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Next, let

$$\vec{w}_2 = \vec{u}_2 - \langle \vec{u}_2, \vec{v}_1 \rangle \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - \left(\frac{1}{3} + 2 + \frac{8}{3} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 5 \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 10/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

And then finally

$$\vec{v}_2 = \frac{1}{\|\vec{w}_2\|}\vec{w}_2 = \frac{1}{1}\begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

So $\left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\}$ is an orthonormal basis for $\operatorname{col}(B)$.

(b) Use your answer from part (a) to compute the orthogonal projection of $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ onto $\mathrm{col}(B)$.

Answer: The projection of $\vec{w}$ onto $\mathrm{col}(B)$ is simply

$$\begin{aligned}
\mathrm{proj}_{\mathrm{col}(B)}\, \vec{w} = \langle \vec{w}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{w}, \vec{v}_2 \rangle \vec{v}_2 &= \left(1 - \frac{2}{3} + \frac{2}{3}\right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + \left(-2 + \frac{1}{3} + \frac{2}{3}\right) \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned}$$