Math 369 Exam #3 Practice Problem Solutions

1. Diagonalize the matrix

\[ A = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}. \]

In other words, find an invertible matrix \( P \) and a diagonal matrix \( D \) so that \( A = PDP^{-1} \).

**Answer:** We need to compute the eigenvalues and eigenvectors of \( A \). For a \( 2 \times 2 \) matrix like this, we can easily just determine the characteristic polynomial and find its roots. The characteristic polynomial is

\[
\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -2 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) + 2 = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4).
\]

Therefore, the eigenvalues are 3 and 4, and now we can find eigenvectors.

The eigenvectors for \( \lambda = 3 \) are solutions to

\[(A - 3I)\vec{v} = \vec{0},\]

so we solve the augmented matrix

\[
\begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \end{pmatrix}.
\]

But now subtracting twice row 1 from row 2 gives

\[
\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

so solutions are the vectors \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) so that \(-x_1 + x_2 = 0\), meaning that \( x_1 = x_2 \). For example, \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is such an eigenvector.

On the other hand, the eigenvectors for \( \lambda = 4 \) are solutions to

\[(A - 4I)\vec{v} = \vec{0},\]

so we solve the augmented matrix

\[
\begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix}.
\]

But now subtracting row 1 from row 2 yields

\[
\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

so solutions are the vectors \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) so that \(-2x_1 + x_2 = 0\), meaning that \( x_1 = 2x_2 \). For example, \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) is such an eigenvector.
Therefore, the eigenvectors form the columns of the matrix

\[
P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
\]

and \( D \) is the diagonal matrix whose nonzero entries are the eigenvalues \( D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \).

2. Let \( A \) be an \( n \times n \) matrix.

(a) Let \( \lambda \in \mathbb{R} \) be a number. Give the definition of the eigenspace \( V_\lambda \) of \( A \) associated to \( \lambda \).

**Answer:** The eigenspace is defined to be

\[ V_\lambda = \{ \vec{v} \in V : A\vec{v} = \lambda\vec{v} \} \].

(b) Show that \( \vec{0} \in V_\lambda \) (this is true regardless of what \( \lambda \) is).

**Proof.** Regardless of what \( A \) is, \( A\vec{0} = \vec{0} = \lambda\vec{0} \), so \( \vec{0} \in V_\lambda \). \( \square \)

(c) Suppose that \( \lambda \neq \tau \), but that \( \vec{v} \in V_\lambda \) and \( \vec{v} \in V_\tau \). Prove that this means \( \vec{v} = \vec{0} \).

**Proof.** Since \( \vec{v} \in V_\lambda \) and \( \vec{v} \in V_\tau \), we know that

\[ A\vec{v} = \lambda\vec{v} \quad \text{and} \quad A\vec{v} = \tau\vec{v}. \]

But since \( A\vec{v} = A\vec{v} \), this means that

\[ \lambda\vec{v} = \tau\vec{v}. \]

In particular, if \( v_i \) is the \( i \)th entry in \( \vec{v} \), this tells us that \( \lambda v_i = \tau v_i \); since \( \lambda \neq \tau \), this can only happen if \( v_i = 0 \). Since the same argument works for all \( i \), this implies that \( \vec{v} = \vec{0} \). \( \square \)

3. Let \( V \) be a vector space with an inner product. Let \( v \) be some particular element of \( V \) and define

\[ W = \{ w \in V : \langle w, v \rangle = 0 \}. \]

Prove that \( W \) is a subspace of \( V \).

**Proof.** As usual, we need to show that \( W \) is closed under addition and scalar multiplication. First, suppose \( w_1, w_2 \in W \). The goal is to show that \( w_1 + w_2 \in W \). Since \( w_1, w_2 \in W \), we know that

\[ \langle w_1, v \rangle = 0 \quad \text{and} \quad \langle w_2, v \rangle = 0. \]

Therefore, using the additivity of the inner product,

\[ \langle w_1 + w_2, v \rangle = \langle w_1, v \rangle + \langle w_2, v \rangle = 0 + 0 = 0, \]
so \( w_1 + w_2 \in W \).

Also, if \( w \in W \) and \( \lambda \in \mathbb{R} \), then we know that \( \langle w, v \rangle = 0 \), so it follows from homogeneity of the inner product that

\[
\langle \lambda w, v \rangle = \lambda \langle w, v \rangle = \lambda \cdot 0 = 0,
\]

so \( \lambda w \in W \).

Therefore, \( W \) is closed under addition and scalar multiplication, so it is a subspace of \( V \). \( \square \)

4. Consider \( \mathbb{R}^3 \) with the slightly unusual inner product

\[
\langle \vec{u}, \vec{v} \rangle = \frac{1}{6} u_1 v_1 + \frac{1}{8} u_2 v_2 + \frac{1}{27} u_3 v_3.
\]

Let \( \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \).

(a) Show that \( \{ \vec{v}_1, \vec{v}_2 \} \) is an orthonormal set with respect to this weird inner product.

**Proof.** We just need to compute \( \langle \vec{v}_1, \vec{v}_1 \rangle \), \( \langle \vec{v}_2, \vec{v}_2 \rangle \), and \( \langle \vec{v}_1, \vec{v}_2 \rangle \):

\[
\langle \vec{v}_1, \vec{v}_1 \rangle = \frac{1}{6} \cdot 1 \cdot 1 + \frac{1}{8} \cdot 2 \cdot 2 + \frac{1}{27} \cdot 3 \cdot 3 = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1
\]

\[
\langle \vec{v}_2, \vec{v}_2 \rangle = \frac{1}{6} \cdot 1 \cdot 1 + \frac{1}{8} \cdot (-2) \cdot (-2) + \frac{1}{27} \cdot 3 \cdot 3 = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1
\]

\[
\langle \vec{v}_1, \vec{v}_2 \rangle = \frac{1}{6} \cdot 1 \cdot 1 + \frac{1}{8} \cdot 2 \cdot (-2) + \frac{1}{27} \cdot 3 \cdot 3 = \frac{1}{6} - \frac{1}{2} + \frac{1}{3} = 0,
\]

so \( \vec{v}_1 \) and \( \vec{v}_2 \) have norm 1 and are orthogonal with respect to this weird inner product. \( \square \)

(b) Let \( \vec{u}_3 = \begin{bmatrix} 0 \\ 8 \\ 18 \end{bmatrix} \). Apply the Gram-Schmidt procedure to the set \( \{ \vec{v}_1, \vec{v}_2, \vec{u}_3 \} \) to get an orthonormal set. (Hint: you already know from part (a) that \( \{ \vec{v}_1, \vec{v}_2 \} \) is orthonormal, so \( \vec{u}_3 \) is the only vector that needs adjustment.)

**Answer:** Since we already have \( \vec{v}_1 \) and \( \vec{v}_2 \), we only need to figure out \( \vec{v}_3 \). Recall that we find \( \vec{v}_3 \) in a two-step process: first, we compute

\[
\vec{w}_3 = \vec{u}_3 - \langle \vec{u}_3, \vec{v}_1 \rangle \vec{v}_1 - \langle \vec{u}_3, \vec{v}_2 \rangle \vec{v}_2
\]

\[
= \begin{bmatrix} 0 \\ 8 \\ 18 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}
= \begin{bmatrix} 0 \\ 8 \\ 18 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}
= \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix}.
\]
Then, we let 
\[ \vec{v}_3 = \frac{1}{\|\vec{w}_3\|} \vec{w}_3 = \frac{1}{\sqrt{4}} \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \]

Therefore, the desired orthonormal set is 
\[ \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \right\} \]

5. Solve the system of differential equations
\[ y_1' = -2y_1 + 2y_2 \]
\[ y_2' = y_1 - y_2. \]

**Answer:** As usual, the strategy is to diagonalize the matrix 
\[ A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \]
as \[ A = PDP^{-1}, \]
solve the simpler system \[ \vec{u}' = D\vec{u}, \]
and then to recover the solution of the original system as \[ \vec{y} = P\vec{u}. \]

So first we compute eigenvalues of \( A \), which are the roots of the characteristic polynomial 
\[ \det(A - \lambda I) = \begin{vmatrix} -2 - \lambda & 2 \\ 1 & -1 - \lambda \end{vmatrix} = (-2 - \lambda)(-1 - \lambda) - 2 = \lambda^2 + 3\lambda + 2 = \lambda^2 + 3\lambda = \lambda(\lambda + 3), \]
so the eigenvalues are \( \lambda = -3 \) and \( \lambda = 0. \)

Now we look for eigenvectors.
\( \lambda = -3: \) We need to solve \((A + 3I)\vec{x} = \vec{0}): 
\[ \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}. \]
It’s clear that row-reducing yields the equation \( x_1 + 2x_2 = 0 \), so \( x_1 = -2x_2 \), and the corresponding eigenvector is \( \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \)
\( \lambda = 0: \) We’re solving \((A - 0I)\vec{x} = \vec{0}, \) or simply \( A\vec{x} = \vec{0}: \)
\[ \begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix}. \]
Adding half of row 1 to row 2 yields 
\[ \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
so our equation is \(-2x_1 + 2x_2 = 0, \) or just \( x_1 = x_2, \) and the eigenvector we’re looking for is \( \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \)
Therefore,

\[ P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}. \]

Therefore, our simplified system is just

\[ \begin{bmatrix} u'_1(x) \\ u'_2(x) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} -3u_1(x) \\ 0 \end{bmatrix}. \]

Therefore, \( u_1(x) = C_1 e^{-3x} \) and \( u_2(x) = C_2 \). Hence, the solutions to our original system are

\[ \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{-3x} \\ C_2 \end{bmatrix} = \begin{bmatrix} -2C_1 e^{-3x} + C_2 \\ C_1 e^{-3x} + C_2 \end{bmatrix}. \]

6. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

(a) Every invertible matrix can be diagonalized.

**Answer:** False. Consider the matrix

\[ A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}. \]

Then \( \det A = 1 \), so \( A \) is indeed invertible. Now, if we try to find the eigenvalues of \( A \), we solve

\[ 0 = \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 5 \\ 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)(1 - \lambda), \]

so \( \lambda = 1 \). Since \( A \) has only one eigenvalue, there must be two linearly independent eigenvectors associated with the eigenvalue 1 – otherwise there won’t be enough columns to make an invertible matrix \( P \).

The eigenvector(s) corresponding to 1 will be in the nullspace of

\[ A - I = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} \]

which is already in reduced echelon form. If \( (A - I)x = 0 \), then it must be the case that \( x_2 = 0 \), so eigenvectors are of the form

\[ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}. \]

Therefore, there are not 2 linearly independent eigenvectors, so \( A \) is not diagonalizable.

(b) Every diagonalizable matrix is invertible.

**Answer:** False. The matrix

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

is certainly diagonalizable (it is itself a diagonal matrix), but \( A \) is not invertible since it is of rank 1 (alternatively: \( \det A = 0 \)).
(c) If the matrix $A$ is not invertible, then 0 is an eigenvalue of $A$.

**Answer:** True. If $A$ is not invertible, then $\text{rank } A < n$. Therefore, since

$$\dim \text{col } A + \dim \text{nul } A = n$$

and $\dim \text{col } A = \text{rank } A$, we see that the nullspace of $A$ has dimension $\geq 1$, so there is at least one non-zero vector $\vec{v}$ such that $A\vec{v} = \vec{0}$. Thus, 0 is an eigenvalue of $A$ with corresponding eigenvector $\vec{v}$.

7. Let

$$B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 2 & 4 & 8 \end{bmatrix}$$

(a) Find an orthonormal basis (using the usual dot product) for the column space of $B$.

**Answer:** First of all, we need to determine some basis for $\text{col}(B)$, so we want to row-reduce to determine the pivot columns of $B$. Subtracting twice row 1 from row 2 and from row 3 yields

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$ 

Now we can subtract twice row 2 from row 3 to get

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Therefore, the first and second columns of $B$ are the pivot columns, so $\{\vec{u}_1, \vec{u}_2\} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is a basis for $\text{col}(B)$. To find an orthonormal basis, we do the Gram-Schmidt process. First, let

$$\vec{v}_1 = \frac{1}{\|\vec{u}_1\|} \vec{u}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}.$$ 

Next, let

$$\vec{w}_2 = \vec{u}_2 - \langle \vec{u}_2, \vec{v}_1 \rangle \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - \left(\frac{1}{3} + 2 + \frac{8}{3}\right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 5/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ -2/3 \end{bmatrix}.$$ 

And then finally

$$\vec{v}_2 = \frac{1}{\|\vec{w}_2\|} \vec{w}_2 = \frac{1}{1} \begin{bmatrix} -2/3 \\ -1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ -2/3 \end{bmatrix}.$$ 

So $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$ is an orthonormal basis for $\text{col}(B)$.
(b) Use your answer from part (a) to compute the orthogonal projection of \( \vec{w} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \) onto \( \text{col}(B) \).

**Answer:** The projection of \( \vec{w} \) onto \( \text{col}(B) \) is simply

\[
\text{proj}_{\text{col}(B)} \vec{w} = \langle \vec{w}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{w}, \vec{v}_2 \rangle \vec{v}_2 = \left( 1 - \frac{2}{3} + \frac{2}{3} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + \left( -2 + \frac{1}{3} + \frac{2}{3} \right) \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \\
= \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \\
= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\]