Math 369 Exam #3 Practice Problems

1. Diagonalize the matrix

\[
A = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}
\]

In other words, find an invertible matrix \( P \) and a diagonal matrix \( D \) so that \( A = PDP^{-1} \).

2. Let \( A \) be an \( n \times n \) matrix.

(a) Let \( \lambda \in \mathbb{R} \) be a number. Give the definition of the eigenspace \( V_\lambda \) of \( A \) associated to \( \lambda \).

(b) Show that \( \vec{0} \in V_\lambda \) (this is true regardless of what \( \lambda \) is).

(c) Suppose that \( \lambda \neq \tau \), but that \( \vec{v} \in V_\lambda \) and \( \vec{v} \in V_\tau \). Show that this means \( \vec{v} = \vec{0} \).

3. Let \( V \) be a vector space with an inner product. Let \( v \in V \) be some particular vector and define \( W = \{ w \in V : \langle w, v \rangle = 0 \} \).

Prove that \( W \) is a subspace of \( V \).

4. Consider \( \mathbb{R}^3 \) with the slightly unusual inner product

\[
\langle \vec{u}, \vec{v} \rangle = \frac{1}{6} u_1 v_1 + \frac{1}{8} u_2 v_2 + \frac{1}{27} u_3 v_3.
\]

Let \( \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \).

(a) Show that \( \{ \vec{v}_1, \vec{v}_2 \} \) is an orthonormal set with respect to this weird inner product.

(b) Let \( \vec{u}_3 = \begin{bmatrix} 0 \\ 8 \\ 18 \end{bmatrix} \). Apply the Gram-Schmidt procedure to the set \( \{ \vec{v}_1, \vec{v}_2, \vec{u}_3 \} \) to get an orthonormal set. (Hint: you already know from part (a) that \( \{ \vec{v}_1, \vec{v}_2 \} \) is orthonormal, so \( \vec{u}_3 \) is the only vector that needs adjustment.)

5. Solve the system of differential equations

\[
\begin{align*}
y_1' &= -2y_1 + 2y_2 \\
y_2' &= y_1 - y_2.
\end{align*}
\]

6. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

(a) Every invertible matrix can be diagonalized.

(b) Every diagonalizable matrix is invertible.

(c) If the matrix \( A \) is not invertible, then 0 is an eigenvalue of \( A \).

7. Let

\[
B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 2 & 4 & 8 \end{bmatrix}
\]

(a) Find an orthonormal basis (using the usual dot product) for the column space of \( B \).

(b) Use your answer from part (a) to compute the orthogonal projection of \( \vec{w} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \) onto \( \text{col}(B) \).