Math 369 Exam #2 Practice Problems

1. Is $\{\begin{bmatrix}1\\-2\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\1\end{bmatrix}, \begin{bmatrix}5\\-8\\1\end{bmatrix}\}$ a basis for $\mathbb{R}^3$?

2. In each part, V is a vector space and S is a subset of V. Determine whether S is a *subspace* of V.

   (a) $V = \mathbb{R}^3$
   $S = \{\begin{bmatrix}x\\12\\3x\end{bmatrix} : x \in \mathbb{R}\}$.

   (b) $V = \mathbb{R}^2$
   $S = \{\begin{bmatrix}x\\y\end{bmatrix} : 2x - 5y = 11\}$

   (c) $V = \mathbb{R}^n$
   $S = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = 2\vec{x}\}$, where A is a particular $n \times n$ matrix.

   (d) $V = F(-\infty, \infty)$
   $S = \{f : f(x) = a\cos x + b\sin x + c\}$

3. Let V be a vector space.

   (a) Define what it means for a set $\{u_1, \ldots, u_n\} \subset V$ to be linearly dependent.

   (b) Suppose $v \in V$. Is the set $\{0, v\}$ linearly dependent? Explain.

   (c) Define what it means for $u \in V$ to be in the span of a set $\{v_1, \ldots, v_n\}$.

   (d) Suppose $\{v_1, \ldots, v_n\}$ is a set of vectors and $u \in \text{span}(v_1, \ldots, v_n)$. Show that $\{v_1, \ldots, v_n, u\}$ is linearly dependent.

4. Let A be a 2×3 matrix.

   (a) Let $U = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0}\}$. Show that U is a subspace of $\mathbb{R}^3$.

   (b) Is $W = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{b}\}$ a subspace when $\vec{b} = \begin{bmatrix}1\\2\end{bmatrix}$? Explain.

5. Let $V = P_\infty$ be the vector space of polynomials. Is the set

$$\{1 + x + x^2, 1 - x, 1 - x^3\}$$

   linearly independent? Prove your claim.

6. Logan and Terry are both computing with the same 5×3 matrix. Logan determines that the nullspace of the matrix is 2-dimensional, while Terry computes that the column space is 2-dimensional. Can they both be right? Justify your answer.

7. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

   (a) If V is a vector space and S is a finite set of vectors in V, then some subset of S forms a basis for V.

   (b) Suppose A is an $m \times n$ matrix such that $A\vec{x} = \vec{b}$ can be solved for any choice of $\vec{b} \in \mathbb{R}^m$. Then the columns of A form a basis for $\mathbb{R}^m$.

   (c) The set of polynomials of degree ≤ 5 forms a vector space.

8. Consider the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & - & 3x_4 & = & b_1 \\ x_1 & + & 2x_2 & + & 2x_3 & - & 5x_4 & = & b_2 \\ 2x_1 & + & 4x_2 & + & 3x_3 & - & 8x_4 & = & b_3 \end{array}.$$

(a) Find all solutions when $b_1 = b_2 = b_3 = 0$. Find a basis for the space of solutions to the homogeneous system.

(b) Let S be the set of vectors $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that the system can be solved. What is the dimension of S?

(c) It's easy to check that the vector $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ is a solution to the system that arises when $b_1 = 3$, $b_2 = 5$, and $b_3 = 8$. Find *all* the solutions to this system.

9. Let $B = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}\}$ and $B' = \{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix}\}$.

(a) Show that B and B' are both bases for $\mathbb{R}^2$.

(b) Find the change-of-basis matrix $[M]_{B \to B'}$ for converting coordinate vectors with respect to B to coordinate vectors with respect to B'.

(c) Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What is the coordinate vector $[\vec{v}]_B$ for $\vec{v}$ with respect to the basis B?

(d) Use your answer to part (b) to determine $[\vec{v}]_{B'}$, the coordinate vector for B with respect to B'.