1. Exercise 3.2.2. Let 
\[ B = \left\{ \frac{(-1)^n}{n+1} : n = 1, 2, 3, \ldots \right\}. \]

(a) Find the limit points of \( B \).

\textit{Answer.} The limit points of \( B \) are 1 and \(-1\). To see that these are limit points, notice that
\[ \left( \frac{2n}{2n+1} \right) \quad \text{and} \quad \left( \frac{2n+1}{2n+2} \right) \]
are sequences contained in \( B \). The first converges to 1 and the second to \(-1\), so these are both limit points.

Now, suppose \( a \in \mathbb{R} \) with \(|a| \neq 1\). Let \( \epsilon = \frac{|a|-1}{2} \). Let \( N > \frac{1}{\epsilon} - 1 \). If \( n \geq N \), then
\[ \left| \frac{(-1)^n}{n+1} - 1 \right| = \left| \frac{n}{n+1} - 1 \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} \leq \frac{1}{N+1} < \epsilon \]
if \( n \) is even and
\[ \left| \frac{(-1)^n}{n+1} - (-1) \right| = \left| \frac{-n}{n+1} + 1 \right| = \left| \frac{1}{n+1} \right| = \frac{1}{n+1} \leq \frac{1}{N+1} < \epsilon \]
if \( n \) is odd. Hence, \( \frac{(-1)^n}{n+1} \) is within \( \epsilon \) of either 1 or \(-1\). Since the distance from \( a \) to both 1 and \(-1\) is at least \( 2\epsilon \), we see that for all \( n \geq N \) the distance from \( a \) to \( \frac{(-1)^n}{n+1} \) is at least \( \epsilon \).

Therefore, at most finitely many elements of \( B \) can be within \( \epsilon \) of either 1 or \(-1\). Since the distance from \( a \) to both 1 and \(-1\) is at least \( 2\epsilon \), we see that for all \( n \geq N \) the distance from \( a \) to \( \frac{(-1)^n}{n+1} \) is at least \( \epsilon \).

Hence, no number \( a \) such that \(|a| \neq 1\) can be a limit point of \( B \), so we conclude that 1 and \(-1\) are the only limit points of \( B \).

(b) Is \( B \) a closed set?

\textit{Answer.} No. As we saw in (a), 1 and \(-1\) are limit points of \( B \). However, neither 1 nor \(-1\) is an element of \( B \), so \( B \) does not contain all of its limit points.

(c) Is \( B \) an open set?

\textit{Answer.} No. Notice that \(-1/2 \in B \). For any \( \epsilon > 0 \), the set
\[ V_{\epsilon}(-1/2) = (-1/2 - \epsilon, -1/2 + \epsilon) \]
contains infinitely many numbers (e.g. \( \min\{-1/2 + \epsilon/2, 0\} \)) that are not elements of \( B \) since no number between \(-1/2 \) and 0 is in \( B \).

(d) Does \( B \) contain any isolated points?

\textit{Answer.} Yes. For example,
\[ V_{1/4}(-1/2) = (-3/4, -1/4) \]
intersects \( B \) only at \(-1/2\), so \(-1/2\) is an isolated point of \( B \). In fact, it’s straightforward to show that \textit{every} element of \( B \) is an isolated point.
(e) Find $\overline{B}$.

**Answer.** By definition, $\overline{B} = B \cup L$ where $L$ is the set of limit points of $B$. We saw in (a) that $L = \{1, -1\}$, so $\overline{B} = B \cup \{1, -1\}$.

2. Exercise 3.2.3. Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no $\epsilon$-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbb{Q}$.

**Answer.** $\mathbb{Q}$ is neither open nor closed. To see that it’s not open, note that $0 \in \mathbb{Q}$, but any $\epsilon$-neighborhood of $0$ contains infinitely many numbers of the form $\sqrt{2}/n$, all of which are irrational. Hence, no $\epsilon$-neighborhood of $0$ is contained in $\mathbb{Q}$.

To see that $\mathbb{Q}$ is not closed, note that every $\epsilon$-neighborhood of $\sqrt{2}$ contains infinitely many rational numbers, so $\sqrt{2}$ is a limit point of $\mathbb{Q}$ that is not contained in $\mathbb{Q}$.

(b) $\mathbb{N}$.

**Answer.** $\mathbb{N}$ is closed and not open. To see that it is not open, notice that, for any $\epsilon > 0$,

$$V_\epsilon(1) = (1 - \epsilon, 1 + \epsilon)$$

contains the number $1 - \epsilon/2 \notin \mathbb{N}$, so $V_\epsilon(1) \nsubseteq \mathbb{N}$, meaning that $\mathbb{N}$ is not open.

(c) $\{x \in \mathbb{R} : x > 0\}$.

**Answer.** This set is open and not closed. To see that it is not closed, notice that, for any $\epsilon > 0$,

$$V_\epsilon(0) = (\epsilon, 0)$$

so $V_\epsilon(0) \cap \{x \in \mathbb{R} : x > 0\} = (0, \epsilon)$, which contains infinitely many numbers (for example, $\epsilon/2 \neq 0$). Therefore, $0$ is a limit point of the set that is not contained in the set.

(d) $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

**Answer.** $(0, 1]$ is not open and not closed. To see that it is not open, notice that, for any $\epsilon > 0$,

$$V_\epsilon(1) = (1 - \epsilon, 1 + \epsilon)$$

contains infinitely many numbers (e.g. $1 + \epsilon/2$) which are not contained in $(0, 1]$. Hence, $V_\epsilon(1) \nsubseteq (0, 1]$ for any $\epsilon$.

On the other hand, for any $\epsilon > 0$,

$$V_\epsilon(0) = (\epsilon, 1)$$

contains infinitely many numbers (e.g. $\epsilon/2$) in the set $(0, 1]$. Hence, $0 \notin (0, 1]$ is a limit point of $(0, 1]$.

(e) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$.

**Answer.** Call this set $S$. $S$ is not open and not closed. To see that $S$ is not open, note that, for any $\epsilon > 0$,

$$V_\epsilon(1) = (1 - \epsilon, 1 + \epsilon)$$
contains infinitely many numbers (e.g. \(1 - \epsilon/2\)) that are not in the set \(S\) (since \(S\) has \(1\) as a lower bound). Hence, \(V_\epsilon(1) \not\subset S\).

To see that \(S\) is not closed, notice that

\[
\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.
\]

Since \(S\) consists of the partial sums of this series, \(\frac{\pi^2}{6}\) must be a limit point of \(S\), yet every element of \(S\) is rational. Hence, \(S\) does not contain all of its limit points.

3. Exercise 3.2.6. Prove Theorem 3.2.8, which says a set \(F \subseteq \mathbb{R}\) is closed if and only if every Cauchy sequence contained in \(F\) has a limit that is also an element of \(F\).

Proof. \((\Rightarrow)\) Suppose \(F \subseteq \mathbb{R}\) is closed. Let \((x_n)\) be a Cauchy sequence such that \(x_n \in F\) for all \(n \in \{1, 2, 3, \ldots\}\). All Cauchy sequences converge, so we know that \((x_n) \to x\) for some \(x \in \mathbb{R}\). If there exists \(n\) such that \(x = x_n \in F\), then we see that \(x \in F\). If \(x \neq x_n\) for all \(n\), then Theorem 3.2.5 implies that \(x\) is a limit point of \(F\) and thus, since \(F\) is closed, \(x \in F\). Either way, we see that every Cauchy sequence has a limit that is also an element of \(F\).

\((\Leftarrow)\) Suppose \(F \subseteq \mathbb{R}\) is a set such that every Cauchy sequence contained in \(F\) has a limit that is also an element of \(F\). Let \(x\) be a limit point of \(F\). Then, by Theorem 3.2.5, there exists a sequence \((x_n)\) such that \(x_n \in F\) and \(x_n \neq x\) for all \(n\) such that \((x_n)\) converges to \(x\). Since \((x_n)\) is convergent, it is a Cauchy sequence and so, by hypothesis, \(x \in F\). Since our choice of limit point \(x\) was arbitrary, we conclude that \(F\) contains all of its limit points, so \(F\) is closed.

4. Exercise 3.2.8. Given \(A \subseteq \mathbb{R}\), let \(L\) be the set of all limit points of \(A\).

(a) Show that the set \(L\) is closed.

Proof. To show that \(L\) is closed, I intend to show that \(L\) contains all of its limit points. To that end, suppose \(x \in \mathbb{R}\) is a limit point of \(L\). My goal is to show that \(x \in L\); since \(L\) is the set of limit points of \(A\), this is equivalent to showing that \(x\) is a limit point of \(A\). Let \(\epsilon > 0\). Then, since \(x\) is a limit point of \(L\), there exists \(y \in L\) with \(y \neq x\) such that

\[
|x - y| < \frac{\epsilon}{2}.
\]

On the other hand, since \(y \in L\), \(y\) is a limit point of \(A\), so there exists \(a \in A\) such that \(a \neq y\) and

\[
|a - y| < |x - y|.
\]

In particular, this implies \(a \neq x\). Therefore,

\[
|a - x| = |a - y + y - x| \leq |a - y| + |y - x| < |x - y| + |x - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Since \(\epsilon > 0\) was arbitrary, this implies that \(x\) is a limit point of \(A\), meaning that \(x \in L\). Since the choice of limit point \(x\) was arbitrary, we conclude that \(L\) contains all of its limit points, so \(L\) is closed. 

\[3\]
(b) Argue that if \( x \) is a limit point of \( A \cup L \), then \( x \) is a limit point of \( A \). Use this observation to furnish a proof for Theorem 3.2.12.

**Proof.** Suppose \( x \) is a limit point of \( A \cup L \). Let \( \epsilon > 0 \). Then since \( x \) is a limit point of \( A \cup L \), there exists \( y \in A \cup L \) such that \( y \neq x \) and

\[
|x - y| < \frac{\epsilon}{2}.
\]

If \( y \in A \), then

\[
|x - y| < \frac{\epsilon}{2} < \epsilon,
\]

so \( x \) is a limit point of \( A \).

If \( y \notin A \), then \( y \in L \). Therefore, \( y \) is a limit point of \( A \), so there exists \( a \in A \) such that

\[
|a - y| < |x - y|.
\]

Therefore, \( a \neq x \) and

\[
|a - x| = |a - y + y - x| \leq |a - y| + |y - x| < |x - y| + |x - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Again, this means that \( x \) is a limit point of \( A \).

Thus, we see that if \( x \) is a limit point of \( A \cup L \), then \( x \) is a limit point of \( A \). Therefore, the set of limit points of \( A \cup L \) is contained in \( L \); in particular, \( A \cup L \) contains all of its limit points, so \( \bar{A} = A \cup L \) is closed, which was all that remained to complete the proof of Theorem 3.2.12. \( \square \)

5. Exercise 3.2.13. Prove that the only sets that are both open and closed are \( \mathbb{R} \) and the empty set \( \emptyset \).

**Proof.** Suppose \( A \) is a set which is both open and closed and that \( A \neq \emptyset \), \( A \neq \mathbb{R} \). Then, since \( A \) is open, \( A^c \) is closed. On the other hand, since \( A \) is closed, \( A^c \) is open. Moreover, since \( A \neq \mathbb{R} \), \( A^c \neq \emptyset \).

Therefore, \( A \) and \( A^c \) are two non-empty sets, each of which is both open and closed. The goal is to see that this leads to a contradiction. Intuitively, the problem is going to arise at the “boundary” between \( A \) and \( A^c \). I want to get at this boundary iteratively.

To that end, let \( a_1 \in A \) and \( b_1 \in A^c \) (\( a_1 \) and \( b_1 \) exist since \( A \) and \( A^c \) are both non-empty) and let \( I_1 = [a_1, b_1] \). Now, let \( m_1 = \frac{a_1 + b_1}{2} \) be the midpoint of the sequence. If \( m_1 \in A \), let \( a_2 = m_1 \) and \( b_2 = b_1 \); otherwise, let \( a_2 = a_1 \) and \( b_2 = m_1 \). Either way, define \( I_2 = [a_2, b_2] \).

In general, let \( m_k = \frac{a_k + b_k}{2} \) be the midpoint of the closed interval \( I_k = [a_k, b_k] \). If \( m_k \in A \), let \( a_{k+1} = m_k \) and let \( b_{k+1} = b_k \); otherwise, let \( a_{k+1} = a_k \) and let \( b_{k+1} = m_k \). Either way, define \( I_{k+1} = [a_{k+1}, b_{k+1}] \).

By construction, \( I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \). By the Nested Interval Property, \( \bigcap_{n=1}^{\infty} I_n \neq \emptyset \), so there exists \( x \in \bigcap_{n=1}^{\infty} I_n \).

**Lemma 1.** \( (a_n) \to x \) and \( (b_n) \to x \).
We’ll prove the lemma in a minute, but first let’s see why this leads directly to a contradiction.

Either \( x \in A \) or \( x \in A^c \). If \( x \in A \), then, since \( A \) is open, there exists \( \epsilon > 0 \) such that \( V_\epsilon(x) \subseteq A \). However, since \( (b_n) \to x \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( b_n \in V_\epsilon(x) \). In particular, \( b_N \in V_\epsilon(x) \subseteq A \), which is clearly impossible since \( b_N \in A^c \). Thus, \( x \notin A \).

Similarly, if \( x \in A^c \), then there exists \( \epsilon > 0 \) such that \( V_\epsilon(x) \subseteq A^c \). But then the fact that \( (a_n) \to x \) means there exists \( N \in \mathbb{N} \) such that \( a_n \in V_\epsilon(x) \) for all \( n \geq N \) and, in particular, \( a_N \in V_\epsilon(x) \subseteq A^c \), which is impossible since \( a_N \in A \). Thus, \( x \notin A^c \). But now the big contradiction is at hand: we’ve just shown that \( x \notin A \) and \( x \notin A^c \), which is definitely impossible because \( x \in \mathbb{R} = A \cup A^c \). Thus, provided we can prove Lemma 1, we can conclude that no such \( A \) exists, and so the only sets which are both open and closed are \( \mathbb{R} \) and \( \emptyset \). \( \square \)

**Proof of Lemma 1.** Let \( d = |a_1 - b_1| \) and let \( \epsilon > 0 \). Pick \( N \in \mathbb{N} \) such that \( N > d/\epsilon + 1 \). For all \( n \geq N \), \( x \in I_n = [a_n, b_n] \), which is an interval of length \( d/2^{n-1} \). Therefore

\[
|a_n - x| \leq \frac{d}{2^{n-1}} \leq \frac{d}{n-1} \leq \frac{d}{N-1} < \epsilon
\]

and

\[
|b_n - x| \leq \frac{d}{2^{n-1}} \leq \frac{d}{n-1} \leq \frac{d}{N-1} < \epsilon.
\]

Therefore, since the choice of \( \epsilon > 0 \) was arbitrary, we see that \( (a_n) \to x \) and \( (b_n) \to x \). \( \square \)