Math 317 HW #5 Solutions

1. Exercise 2.4.2.

(a) Prove that the sequence defined by \(x_1 = 3\) and
\[x_{n+1} = \frac{1}{4 - x_n}\]
converges.

Proof. I intend to use the Monotone Convergence Theorem, so my goal is to show that \((x_n)\) is decreasing and bounded. To do so, I will prove by induction that, for any \(n \in \{1, 2, 3, \ldots\}\),
\[0 < x_{n+1} < x_n < 4.\]

Base Case When \(n = 1\), we have that \(x_1 = 3\) and \(x_2 = \frac{1}{4-3} = 1\), so
\[0 < x_2 < x_1.\]

Inductive Step Suppose \(0 < x_{k+1} < x_k < 4\); our goal is to show that this implies that
\[0 < x_{k+2} < x_{k+1} < 4.\]

Since \(x_{k+1} < x_k\), we know that
\[4 - x_{k+1} > 4 - x_k\]
or, equivalently,
\[x_{k+2} = \frac{1}{4 - x_{k+1}} < \frac{1}{4 - x_k} = x_{k+1}.\]

Moreover, since \(x_{k+1} < x_k < 4\), we have
\[x_{k+2} < x_{k+1} < 4.\]

The fact that \(x_{k+1} < 4\) also implies that \(4 - x_{k+1} > 0\), so
\[0 < \frac{1}{4 - x_{k+1}} = x_{k+2}.\]

Putting this all together, then, we see that \(0 < x_{k+1} < x_k < 4\) implies that \(0 < x_{k+2} < x_{k+1} < 4\).

Therefore, by induction, we can conclude that
\[0 < x_{n+1} < x_n < 4\]
for all \(n \in \{1, 2, 3, \ldots\}\). Thus, the sequence \((x_n)\) is a bounded, decreasing sequence and so, by the Monotone Convergence Theorem, it must converge. \qed
(b) Now that we know \( \lim x_n \) exists, explain why \( \lim x_{n+1} \) must also exist and equal the same value.

*Answer.* Intuitively, \( \lim x_{n+1} \) converges because \( (x_{n+1}) \) is identical to \( (x_n) \) except for the very first term, which has no impact on the limiting behavior of the sequence.

If I want to be really rigorous, here’s how to do that: define the sequence \( (y_n) \) by \( y_n = x_{n+1} \) for all \( n \in \{1, 2, 3, \ldots \} \). Let \( \epsilon > 0 \). Since \( (x_n) \) converges to, say, \( L \in \mathbb{R} \), there exists an \( N \in \mathbb{N} \) such that, if \( n \geq N \),

\[
|x_n - L| < \epsilon.
\]

Therefore, if \( n \geq N - 1 \),

\[
|y_n - L| = |x_{n+1} - L| < \epsilon
\]

since \( n + 1 \geq N \). Therefore, since the choice of \( \epsilon > 0 \) was arbitrary, we conclude that \((y_n) \to L \) or, equivalently, \((x_{n+1}) \to L \).

(c) Take the limit of each side of the recursive equation in part (a) of this exercise to implicitly compute \( \lim x_n \).

*Answer.* On the one hand, if \( (x_n) \to L \), we know that \( (x_{n+1}) \to L \). On the other hand, we know

\[
(x_{n+1}) = \left( \frac{1}{4 - x_n} \right) \to \frac{1}{4 - L}
\]

by the Algebraic Limit Theorem (we know \( L \neq 4 \) since 3 is an upper bound for \( (x_n) \)).

Therefore, it must be the case that

\[
L = \frac{1}{4 - L}
\]

or, equivalently,

\[
L(4 - L) = 1.
\]

We can re-write this as

\[
0 = L^2 - 4L + 1,
\]

so, by the quadratic formula,

\[
L = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}.
\]

Since \( 2 + \sqrt{3} > 3 \), which is an upper bound for \( (x_n) \), we know \( L \neq 2 + \sqrt{3} \). Therefore, we can conclude that \( L = 2 - \sqrt{3} \approx 0.268 \).

2. Exercise 2.4.6. Let \( (a_n) \) be a bounded sequence.

(a) Prove that the sequence defined by \( y_n = \sup \{ a_k : k \geq n \} \) converges.

*Proof.* For any \( n \in \{1, 2, 3, \ldots \} \), define the set

\[
A_n := \{a_n, a_{n+1}, a_{n+2}, \ldots \}.
\]
In other words, \( y_n = \sup A_n \). As defined, \( A_{n+1} \subseteq A_n \), so

\[
y_{n+1} \leq y_n.
\]

Since this is true for any \( n \), we see that the sequence \( (y_n) \) is decreasing. Also, since \( (a_n) \) is bounded, there exists \( M > 0 \) such that \( |a_n| \leq M \) for all \( n \in \{1, 2, 3, \ldots\} \). In particular, for any \( n \),

\[
-M \leq a_n \leq y_n \leq y_1,
\]

so we see that the sequence \( (y_n) \) is bounded. Therefore, since \( (y_n) \) is a bounded, decreasing sequence, the Monotone Convergence Theorem implies that it converges.

\( \square \)

(b) The limit superior of \( (a_n) \), or \( \lim \sup a_n \), is defined by

\[
\lim \sup a_n = \lim y_n,
\]

where \( y_n \) is the sequence from part (a) of this exercise. Provide a reasonable definition for \( \lim \inf a_n \) and briefly explain why it always exists for any bounded sequence.

\textit{Answer.} Here’s the definition of \( \lim \inf a_n \):

For a bounded sequence \( (a_n) \), define the sequence \( (z_n) \) by

\[
z_n = \inf \{a_k : k \geq n\}.
\]

Then the limit inferior of \( (a_n) \) is defined to be

\[
\lim \inf a_n := \lim z_n.
\]

The sequence \( (z_n) \) is increasing and bounded (it’s bounded below by \( z_1 \) and above by the upper bound for \( (a_n) \)), so the Monotone Convergence Theorem implies it converges, so the above limit definitely exists.

(c) Prove that \( \lim \inf a_n \leq \lim \sup a_n \) for every bounded sequence, and give an example of a sequence for which the inequality is strict.

\textit{Proof.} Note that, with \( (y_n) \) and \( (z_n) \) defined as above,

\[
z_n \leq a_n \leq y_n
\]

for all \( n \in \{1, 2, 3, \ldots\} \). In particular, \( z_n \leq y_n \). Therefore, by Theorem 2.3.4(ii),

\[
\lim \inf a_n = \lim z_n \leq \lim y_n = \lim \sup a_n.
\]

\( \square \)

Consider the sequence \( (a_n) \) given by

\[
a_n = 2 + (-1)^n(1 + 1/n).
\]

Then it’s straightforward to show that \( \lim \inf a_n = 1 \) and \( \lim \sup a_n = 3 \), so the inequality is strict for this sequence.
(d) Show that $\lim \inf a_n = \lim \sup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. To show the forward implication, assume $\lim \inf a_n = \lim \sup a_n$. In other words, with $(y_n)$ and $(z_n)$ defined as above, $\lim z_n = \lim y_n$. Now, we know that

$$z_n \leq a_n \leq y_n$$

for all $n \in \{1, 2, 3, \ldots\}$, so the Squeeze Theorem implies that $\lim a_n$ exists and is equal to the common limit of the $y_n$’s and $z_n$’s.

Turning to the reverse implication, assume $\lim a_n = L$ for some $L \in \mathbb{R}$. Let $\epsilon > 0$. Since $(a_n) \to L$, there exists $N \in \mathbb{N}$ such that, whenever $n \geq N$,

$$|a_n - L| < \epsilon/2$$

or, equivalently,

$$-\epsilon/2 < a_n - L < \epsilon/2.$$ 

Hence, $L - \epsilon/2 < a_n < L + \epsilon$ for any $n \geq N$, so $L + \epsilon$ is an upper bound for the set $A_n = \{a_k : k \geq n\}$.

Since $y_n = \sup A_n$, this implies that $y_n \leq L + \epsilon/2$, meaning that $y_n - L \leq \epsilon/2$. On the other hand, $y_n \geq a_n > L - \epsilon/2$.

Putting this all together, we see that, for any $n \geq N$,

$$L - \epsilon/2 \leq y_n \leq L + \epsilon/2,$$

which implies that

$$|y_n - L| \leq \epsilon/2 < \epsilon,$$

Since our choice of $\epsilon > 0$ was arbitrary, we conclude that $(y_n) \to L$.

A totally equivalent argument shows that $(z_n) \to L$ as well, so we have that $\lim \inf a_n = \lim \sup a_n = \lim a_n$ when $(a_n)$ converges.

Having proved both implications, we can conclude that $\lim \inf a_n = \lim \sup a_n$ if and only if $\lim a_n$ exists. $\square$

3. Exercise 2.5.3. Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

Example. Consider the sequence $(a_n)$ given by

$$a_n = \begin{cases} 
1 + 1/n & \text{if } n \text{ is odd} \\
1/n & \text{if } n \text{ is even}.
\end{cases}$$

Clearly, $(1 + 1/n) \to 1$ and $(1/n) \to 0$, so the subsequence of odd terms converges to 1 and the subsequence of even terms converges to 0, even though neither 1 nor 0 appear as terms of the sequence.
(b) A monotone sequence that diverges but has a convergent subsequence.

This is impossible. To see why, suppose \((a_n)\) is a monotone sequence with a convergent subsequence \((a_{n_k})\). In fact, for simplicity, assume \((a_n)\) is increasing. Since \((a_{n_k})\) converges, it must be bounded, so there exists \(M \in \mathbb{R}\) such that \(a_{n_k} \leq M\) for all of the \(a_{n_k}\). Now, for any term \(a_n\) from the original sequence, there exists \(k \in \mathbb{N}\) such that \(n_k > n\); then

\[
a_1 \leq a_n \leq a_{n_k} \leq M.
\]

Since our choice of \(n\) was arbitrary, we see that \(a_1 \leq a_n \leq M\) for all \(n \in \{1, 2, 3, \ldots\}\), so the sequence \((a_n)\) must be bounded. A similar argument works when \((a_n)\) is decreasing.

Therefore, any monotone sequence with a convergent subsequence must be bounded and, therefore, convergent by the Monotone Convergence Theorem.

(c) A sequence that contains subsequences converging to every point in the infinite set \(\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}\).

Example. Consider the sequence

\[
(1, 1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, 1, 1/2, 1/3, 1/4, 1/5, 1, \ldots).
\]

In other words, I’m constructing the sequence by stringing together blocks of the form

\[
1, 1/2, 1/3, \ldots, 1/n,
\]

starting with the block corresponding to \(n = 1\), then moving on to the block corresponding to \(n = 2\), etc.

Clearly, each number \(1/k\) appears in the above sequence infinitely many times, which gives a subsequence converging to \(1/k\).

(d) An unbounded sequence with a convergent subsequence.

Example. Consider the sequence \((a_n)\) given by

\[
a_n = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
n & \text{if } n \text{ is even}.
\end{cases}
\]

Then, for any \(M > 0\), there exists \(N \in \mathbb{N}\) such that \(N > M\) (by the Archimedean Property); then either \(a_N = N > M\) or \(a_{N+1} = N + 1 > M\). Since this is true for any \(M > 0\), we see that \((a_n)\) is unbounded.

On the other hand, the subsequence of odd terms is just the sequence \((0, 0, 0, 0, \ldots)\), which clearly converges to 0.

(e) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

This is impossible. To see why, suppose \((a_n)\) has a bounded subsequence \((a_{n_k})\). Then \((a_{n_k})\) is a bounded sequence and so, by the Bolzano-Weierstrass theorem, contains a convergent subsequence \((a_{n_{k'}})\). But, of course, the sub-subsequence \((a_{n_{k'}})\) is also a subsequence of the original sequence \((a_n)\), so \((a_n)\) must contain a convergent subsequence.

4. Exercise 2.5.4. Assume \((a_n)\) is a bounded sequence with the property that every convergent subsequence of \((a_n)\) converges to the same limit \(a \in \mathbb{R}\). Show that \((a_n)\) must converge to \(a\).
Proof. I’ll prove the contrapositive. In other words, I want to show that, if \((a_n)\) is a bounded sequence that does not converge to the number \(a \in \mathbb{R}\), then not every convergent subsequence converges to \(a\) (or, more positively, there is some subsequence that converges to \(b \neq a\)).

To that end, suppose \((a_n)\) is a bounded sequence that does not converge to \(a\). In other words, it is not the case that, for every \(\epsilon > 0\), there is an \(N \in \mathbb{N}\) such that, for all \(n \geq N\), \(|a_n - a| < \epsilon\). Passing the negation through the various quantifiers, this means that there is some \(\epsilon_0 > 0\) such that, for all \(N \in \mathbb{N}\), there is an \(n \geq N\) such that \(|a_n - a| \geq \epsilon_0\). Stated in plainer language, we’re assuming that there is some \(\epsilon_0 > 0\) such that infinitely many of the \(a_n\’s\) are at a distance at least \(\epsilon_0\) away from \(a\).

Now, these infinitely many \(a_n\’s\) form a subsequence; call it \((a_{n_k})\). Since \((a_n)\) is bounded, the subsequence \((a_{n_k})\) is also bounded and thus, by the Bolzano-Weierstrass Theorem, contains a convergent subsequence \((a_{n_{k_i}}) \to b\). Since each of the \(a_{n_{k_i}}\) is at a distance at least \(\epsilon_0\) from \(a\), we definitely have that \(b \neq a\), so we see that not every convergent subsequence of \((a_n)\) converges to \(a\), which is what we wanted to show.

5. Exercise 2.6.2. Supply a proof for Theorem 2.6.2, which says that every convergent sequence is a Cauchy sequence.

Proof. Suppose the sequence \((a_n)\) converges to \(a\). Let \(\epsilon > 0\). Then, by the definition of convergence, there exists \(N \in \mathbb{N}\) such that, for any \(n \geq N\),

\[|a_n - a| < \epsilon/2.\]

Therefore, for any \(n, m \geq N\), we have that

\[|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| < \epsilon/2 + \epsilon/2 = \epsilon,\]

where the first inequality is just an application of the triangle inequality and the second follows from the assumption on \(n\) and \(m\).

Therefore, since our choice of \(\epsilon > 0\) was arbitrary, we see that for all \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that, if \(n, m \geq N\), \(|a_n - a_m| < \epsilon\), which means that \((a_n)\) is a Cauchy sequence. \(\square\)