

## Solutions to Self-Scheduled Part of Math 317 Exam #1

1. (8 points) For each of the following statements, say whether it is true or false. Please write “True” or “False” and not just “T” or “F” (since these letters are easily mistaken for each other). If the statement is true give a brief explanation why (this does not need to be a proof); if it is false, give a counterexample.

- (a) Every bounded set contains its supremum.

*Answer.* False. Consider the set  $A = (0, 1)$ .  $A$  is clearly bounded above and below, as 0 is a lower bound and 1 is an upper bound. Moreover,  $1 = \sup A$ , but  $1 \notin A$ , so  $A$  is an example of a bounded set that does not contain its supremum.

- (b) A decreasing sequence is always bounded.

*Answer.* False. The sequence  $(0, -1, -2, -3, \dots)$  is a decreasing sequence that is unbounded (since it has no lower bound).

- (c) Every Cauchy sequence is bounded.

*Answer.* True. Every Cauchy sequence converges (Theorem 2.6.2) and every convergent sequence is bounded (Theorem 2.3.2). Alternatively, Lemma 2.6.3 says Cauchy sequences are bounded.

- (d) Every sequence contains a convergent subsequence.

*Answer.* False. The sequence  $(1, 2, 3, \dots)$  contains no convergent subsequences since every subsequence is unbounded.

2. (6 points) Give examples of each of the following, or explain why no such example exists (this explanation does not need to be a proof, but you should give a cogent explanation).

- (a) An alternating sequence that converges.

*Answer.* The sequence  $(a_n)$  given by

$$a_n = \frac{(-1)^n}{n}$$

is an alternating sequence that converges to zero.

- (b) A bounded, divergent sequence.

*Answer.* The sequence  $(a_n)$  given by

$$a_n = (-1)^n$$

is a bounded sequence that diverges.

3. (4 points) Determine the following numbers (don't worry about proofs; just find the answer any way you like).

- (a) The infimum of the set

$$\{x \in \mathbb{R} : -1 < 2x - x^2\}.$$

*Answer.* If  $x$  is in the given set,  $-1 < 2x - x^2$  or, equivalently,

$$-x^2 + 2x + 1 > 0.$$

Hence, the extrema of the set are given by those values of  $x$  such that  $-x^2 + 2x + 1 = 0$ . Using the quadratic formula, the extrema are

$$x = \frac{-2 \pm \sqrt{4 - 4(-1)(1)}}{2(-1)} = \frac{-2 \pm \sqrt{8}}{-2} = 1 \mp \sqrt{2}.$$

The smaller of the two is the infimum, so we see that the infimum of the set is  $1 - \sqrt{2}$ .

- (b)  $\lim_{n \rightarrow \infty} a_n$  where the terms of the sequence  $(a_n)$  are given by

$$a_n = \frac{2n - 3}{5n}.$$

*Answer.* The limit of the sequence is  $2/5$ . For a rigorous proof (which you didn't need to give), let  $\epsilon > 0$  and choose  $N \in \mathbb{N}$  such that  $N > \frac{3}{5\epsilon}$ . Then, for any  $n \geq N$ ,

$$n \geq N > \frac{3}{5\epsilon},$$

so

$$\begin{aligned} \left| a_n - \frac{2}{5} \right| &= \left| \frac{2n - 3}{5n} - \frac{2}{5} \right| = \left| \frac{2n - 3}{5n} - \frac{2n}{5n} \right| \\ &= \left| \frac{-3}{5n} \right| \\ &= \frac{3}{5n} \\ &< \frac{3}{5 \cdot \frac{3}{5\epsilon}} \\ &= \epsilon. \end{aligned}$$

Since our choice of  $\epsilon > 0$  was arbitrary, we see that  $\lim a_n = 2/5$ .

**Extra Credit Problem** (3 points) Determine the cardinality of the set of convergent subsequences of the sequence

$$(0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, \dots).$$

*Answer.* The set of convergent subsequences of this sequence has the same cardinality as the real numbers. This result is a consequence of the following:

**Lemma 0.1.** *The set of subsequences of a sequence has the same cardinality as the real numbers.*

To see why the result follows, note that the given sequence has at least one convergent subsequence, namely the sequence of odd terms  $(0, 0, 0, \dots)$ . Then the lemma implies that this subsequence has uncountably many subsequences of its own, all of which converge and all of which are subsequences of the original sequence. The only worry is that there might be even more convergent subsequences, but the lemma implies that the total number of subsequences (convergent or not) of the original sequence also has the cardinality of the real numbers.

*Proof of Lemma.* Let  $(a_n)$  be a sequence. Then any infinite subset of  $\mathbb{N}$  (thought of here as  $\{1, 2, 3, \dots\}$ ) determines a subsequence of  $(a_n)$ . For example, the infinite subset  $\{1, 2, 5, 7, 32, 34, \dots\}$  determines the subsequence

$$(a_1, a_2, a_5, a_7, a_{32}, a_{34}, \dots).$$

On the other hand, the indices of any subsequence clearly form an infinite subset of  $\mathbb{N}$ . Therefore, the set of subsequences is in bijective correspondence with the set of infinite subsets of  $\mathbb{N}$ . You showed in HW #3, Problem 3, that the set of all finite subsets of  $\mathbb{N}$  is countable. Since the power set of  $\mathbb{N}$  is the union of the set of finite subsets and the set of infinite subsets, and since  $\mathcal{P}(\mathbb{N})$  has the same cardinality as  $\mathbb{R}$ , this implies that the set of infinite subsets of  $\mathbb{N}$  has the same cardinality as  $\mathbb{R}$ . Therefore, the set of subsequences of  $(a_n)$  has the same cardinality as  $\mathbb{R}$ .  $\square$