Solutions to Take-Home Part of Math 317 Exam #2

1. Suppose $S \subseteq \mathbb{R}$ is connected and contains more than 1 point. Show that every element of S is a limit point of S.

   Proof. Let $x \in S$; the goal is to show that x is a limit point of S. Define $A := \{x\}$ and $B := S \backslash A$. Since S contains more than one element, B is nonempty. Then certainly A and B are disjoint and $A \cup B = S$. Since S is connected, we know that A and B are not separated, so either $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

   Since A has no limit points, A is closed, so $\overline{A} = A$, and hence $\overline{A} \cap B = A \cap B = \emptyset$. Therefore, it must be the case that $A \cap \overline{B} \neq \emptyset$. Since $A = \{x\}$, this means that $A \cap \overline{B} = \{x\}$, so $x \in \overline{B}$, meaning that x is a limit point of B. This, in turn, implies that x is a limit point of S, as desired.

   Since our choice of $x \in S$ was arbitrary, we conclude that every element of S is a limit point. $\square$

2. Let $A \subseteq \mathbb{R}$ be uncountable. Show that A has a limit point.

   Proof. I will prove the contrapositive, that if A has no limit points then A is either finite or countable. Suppose $A \subseteq \mathbb{R}$ has no limit points. For each $n = 1, 2, 3, \ldots$, let $A_n = [-n, n] \cap A$. Note that A_n is bounded and, since A has no limit points, A_n cannot have any limit points. Therefore, Lemma 2.1, stated and proved below, implies that A_n must be finite. Now,

   $$\bigcup_{n=1}^{\infty} A_n = A.$$

   . Since each A_n is finite and since countable unions of finite sets are either finite or countable, we see that A cannot be uncountable. Thus, I've shown that if a set has no limit points then it cannot be uncountable, as desired. $\square$

   Lemma 2.1. *Let $B \subseteq \mathbb{R}$ be a bounded set. If B is infinite then B has a limit point.*

Proof. Suppose B is infinite. Then I can form a sequence (b_n) contained in B such that $b_i \neq b_j$ whenever $i \neq j$. Since (b_n) is a bounded sequence, the Bolzano–Weierstrass Theorem implies that it contains a convergent subsequence (b_{n_k}). Let L be the limit of (b_{n_k}). Since the b_{n_k} are all distinct, at most one term can be equal to L. Form the subsequence $(b_{n_{k_\ell}})$ from (b_{n_k}) by deleting the term equal to L if it exists.

Thus, $(b_{n_{k_\ell}})$ is a sequence contained in B and converging to L with $b_{n_{k_\ell}} \neq L$ for all ℓ. Therefore, L is a limit point of B, so B does indeed have a limit point. □

3. Suppose $A, B \subseteq \mathbb{R}$ with A compact, B closed, and $A \cap B = \emptyset$.

   (a) Show that there exists $\epsilon > 0$ such that $|a - b| > \epsilon$ for all $a \in A$ and $b \in B$.

   Proof. Define the set

   $$S = \{|a - b| : a \in A, b \in B\}.$$

   Since $A \cap B = \emptyset$, $|a - b| > 0$ for all $a \in A$ and $b \in B$, so S is bounded below (by 0), and so has a greatest lower bound $s = \inf S$. In other words, $|a - b| \geq s$. If I can show that $s > 0$, then I will have that $|a - b| > \epsilon$ for $\epsilon = s/2 > 0$, as desired.

   To see that $s > 0$, assume, for the sake of contradiction, that $s = 0$. By the version of Lemma 1.3.7 appropriate for infima, for all $\alpha > 0$ there exists $x \in S$ such that $x < 0 + \alpha = \alpha$. In particular, for all $n = 1, 2, 3, \ldots$, there exists $s_n \in S$ such that $s_n < 1/n$. Note that

   $$0 \leq s_n < 1/n,$$

   so the sequence (s_n) converges to 0.

   For each n there exist $a_n \in A$ and $b_n \in B$ such that $|a_n - b_n| = s_n$. Since A is compact, the sequence (a_n) contains a subsequence (a_{n_k}) converging to $a \in A$.

   Let $\alpha > 0$. Then, since $(a_{n_k}) \to a$, there exists $N_1 \in \mathbb{N}$ such that, if $k \geq N_1$,

   $$|a_{n_k} - a| < \alpha/2.$$

   On the other hand, since $(s_n) \to 0$, the subsequence (s_{n_k}) also converges to 0 and so there exists $N_2 \in \mathbb{N}$ such that, if $k \geq N_2$, then

$$|s_{n_k}| < \alpha/2.$$

Note that $|s_{n_k}| = s_{n_k}$ since $s_{n_k} \geq 0$ for all k.
If $N = \max\{N_1, N_2\}$ and $k \geq N$, then

$$\begin{aligned} |b_{n_k} - a| = |b_{n_k} - a_{n_k} + a_{n_k} - a| &\leq |b_{n_k} - a_{n_k}| + |a_{n_k} - a| \\ &= s_{n_k} + |a_{n_k} - a| \\ &< \alpha/2 + \alpha/2 \\ &= \alpha. \end{aligned}$$

Therefore, since the choice of $\alpha > 0$ was arbitrary, the sequence (b_{n_k}) converges to a. Since $b_{n_k} \neq a$ for all k, this implies that a is a limit point of B. In turn, since B is closed, this means that $a \in B$. However, this is impossible since $a \in A$ and $A \cap B = \emptyset$.
From this contradiction, then, I can conclude that $s > 0$. □

(b) Does the result in (a) still hold when A is closed but not necessarily compact? (B is still closed.)
Answer. No. Consider the sets

$$A = \{n : n = 2, 3, 4, \ldots\} \quad \text{and} \quad B = \{n + 1/n : n = 2, 3, 4, \ldots\}.$$

Both A and B consist entirely of isolated points (in each case, the distance between distinct elements is at least $1/2$), so both A and B are closed.
For any $\epsilon > 0$, the Archimedean Property implies that there exists $N \in \mathbb{N}$ such that $N > \max\{1/\epsilon, 1\}$. Now, $N \in A$ and $N + 1/N \in B$ and

$$|N - (N + 1/N)| = |-1/N| = 1/N < \epsilon.$$

Since such an N exists for every $\epsilon > 0$, we see that there is no $\epsilon > 0$ such that $|a - b| > \epsilon$ for all $a \in A$ and $b \in B$.

4. Suppose $A \subseteq \mathbb{R}$ and $a \in A$. Then a is called an *interior point* of A if there exists some $\epsilon > 0$ such that $V_\epsilon(a) \subseteq A$. The *interior* of A is defined to be the set $\mathring{A}$ consisting exactly of the interior points of A.

   (a) If $A \subseteq \mathbb{R}$, do A and $\overline{A}$ always have the same interior? Do A and $\mathring{A}$ always have the same closure?

Answer. No, A and $\overline{A}$ do *not* always have the same interior. Consider the set $A = (-\infty, 0) \cup (0, +\infty)$. Then $\mathring{A} = A$. On the other hand, $\overline{A} = \mathbb{R}$, so the interior of $\overline{A}$ is all of $\mathbb{R}$.

It is also not true that A and $\mathring{A}$ have the same closure. Consider the set $A = \{1\}$. There is no $\epsilon > 0$ such that $V_\epsilon(1) \subseteq A$, so A has no interior points. Hence, $\mathring{A} = \emptyset$, so the closure of $\mathring{A}$ is also the empty set. On the other hand, A is already closed, so $\overline{A} = A \neq \emptyset$.

(b) Prove that $\mathring{A}$ is always open. What is the complement of $\mathring{A}$?

Proof. If $\mathring{A} = \emptyset$, then $\mathring{A}$ is clearly open. If $\mathring{A} \neq \emptyset$, let $a \in \mathring{A}$. Then a is an interior point of A, so there exists $\epsilon > 0$ such that $V_\epsilon(a) \subseteq A$. I claim that, in fact, $V_\epsilon(a) \subseteq \mathring{A}$, so I need to show that every element of $V_\epsilon(a)$ is an interior point of A.

To see this, suppose $x \in V_\epsilon(a) = (a - \epsilon, a + \epsilon)$; in other words,

$$a - \epsilon < x < a + \epsilon.$$

Let $\delta = \min\{x - (a - \epsilon), (a + \epsilon) - x\}$. Then $\delta > 0$ and, by construction, $V_\delta(x) \subseteq V_\epsilon(a) \subseteq A$. Hence, x is an interior point of A, so $x \in \mathring{A}$. Since the choice of $x \in V_\epsilon(a)$ was arbitrary, we conclude that $V_\epsilon(a) \subseteq \mathring{A}$. In turn, since the choice of $a \in \mathring{A}$ was arbitrary, we see that every element of $\mathring{A}$ has such an ϵ-neighborhood contained in $\mathring{A}$, so $\mathring{A}$ is open. □

For the second part of the question, I claim that the complement of $\mathring{A}$ is equal to the closure of the complement of A.

Proof. Let x be in the complement of $\mathring{A}$. Then x is not an interior point of A, meaning that for every $\epsilon > 0$ the set $V_\epsilon(x)$ is not contained in A. Said another way, for every $\epsilon > 0$ the set $V_\epsilon(x)$ intersects A^c somewhere other than x, so x is a limit point of A^c. Thus, x is in the closure of A^c. Since the choice of x was arbitrary, we see that the complement of $\mathring{A}$ is contained in the closure of A^c.

On the other hand, if x is in the closure of A^c, then either $x \in A^c$ or x is a limit point of A^c (or both). Either way, for all $\epsilon > 0$ the set $V_\epsilon(x) \cap A^c$ is nonempty, so x is not an interior point of A. Hence, x is in the complement of $\mathring{A}$. Since the choice of x was arbitrary, we see that the closure of A^c is contained in the complement of $\mathring{A}$.

Having proved containment both ways, we can conclude that the complement of $\mathring{A}$ is equal to the closure of A^c, as desired. $\square$

5. (a) Give a rigorous definition (in the style of Definition 4.2.1) of

$$\lim_{x \to +\infty} f(x) = L.$$

Definition 5.1. If $f : A \to \mathbb{R}$ and the set A is not bounded above, then we say that

$$\lim_{x \to +\infty} f(x) = L$$

if, for all $\epsilon > 0$ there exists $M > 0$ such that $x > M$ and $x \in A$ implies that

$$|f(x) - L| < \epsilon.$$

(b) Using your definition from part (a), show that

$$\lim_{x \to +\infty} \frac{1}{x^2} = 0.$$

Proof. Let $\epsilon > 0$. Choose $M = 1/\sqrt{\epsilon}$. Then for any $x > M$ we have that

$$\left| \frac{1}{x^2} - 0 \right| = \left| \frac{1}{x^2} \right| = \frac{1}{x^2} < \epsilon.$$

Since the choice of $\epsilon > 0$ was arbitrary, we see that there is such an M for all $\epsilon > 0$, so

$$\lim_{x \to \infty} \frac{1}{x^2} = 0.$$

$\square$