# Solutions to Take-Home Part of Math 317 Exam #1

1. Suppose $(x_n)$ is a bounded sequence with

$$\liminf x_n = a \quad \text{and} \quad \limsup x_n = b.$$

Show that there exists a subsequence of $(x_n)$ converging to $a$ and a subsequence converging to $b$.

*Proof.* For each $m \in \mathbb{N}$, let $y_m = \sup\{x_m, x_{m+1}, \ldots\}$. Then, by Lemma 1.3.7, there exists $n_m \geq m$ such that

$$y_m - \frac{1}{m} < x_{n_m}.$$

Equivalently, since $x_{n_m} \leq y_m$, we know that

$$|x_{n_m} - y_m| = y_m - x_{n_m} < \frac{1}{m}. \tag{1}$$

The goal is to show that the subsequence $(x_{n_m})$ converges to $b$.

Let $\epsilon > 0$. Since $\lim y_m = \limsup x_n = b$, we know there exists $N_1 \in \mathbb{N}$ such that, for all $m \geq N_1$,

$$|y_m - b| < \frac{\epsilon}{2}. \tag{2}$$

Therefore, for $m \geq \max\{2/\epsilon, N_1\}$,

$$|x_{n_m} - b| = |x_{n_m} - y_m + y_m - b| \leq |x_{n_m} - y_m| + |y_m - b| < \frac{1}{m} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where the first inequality is an application of the triangle inequality, the second follows from (1) and (2), and the third from the choice of $m$. Therefore, since our choice of $\epsilon > 0$ was arbitrary, we conclude that the subsequence $(x_{n_m}) \to b$.

A similar argument using the sequence $(z_m)$ given by $z_m = \inf\{x_m, x_{m+1}, \ldots\}$ and the version of Lemma 1.3.7 suitable for infima (see Exercise 1.3.2, which you did as a warm-up for HW #2, Problem 1) yields a subsequence converging to $a$. □

2. In class, we used the Axiom of Completeness (via the Nested Interval Property) to prove the Bolzano–Weierstrass Theorem. For this problem, do the opposite: use the Bolzano–Weierstrass Theorem to prove the Axiom of Completeness.

*Proof.* This will follow in two parts.

**Lemma 0.1.** *The Bolzano–Weierstrass Theorem implies the Nested Interval Property.*

*Proof.* Let $I_n = [a_n, b_n]$ for each $n$ so that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$. For each $n$,

$$a_1 < b_n \leq b_1,$$

so the sequence $(b_n)$ is bounded. By the Bolzano–Weierstrass Theorem, it contains a convergent subsequence $(b_{n_k}) \to b$.

I claim that $b \leq b_n$ for all $n$. To see this, note that $(b_n)$ and $(b_{n_k})$ are both decreasing sequences since the intervals are nested. If $b > b_m$ for some $m$, then for all $k \geq m$ we have $n_k \geq n_m \geq m$, so $b_{n_k} \leq b_{n_m} \leq b_m < b$ and

$$|b_{n_k} - b| = b - b_{n_k} \geq b - b_m.$$

This is clearly impossible since $b - b_m$ is a fixed positive number and the sequence $(b_{n_k})$ converges to $b$. Therefore, $b \leq b_m$ for all $m$.

On the other hand, $b \geq a_n$ for all $n$. This follows because, for any $n$, $b_{n_k} \geq a_n$, so, by the Order Limit Theorem, $b = \lim b_{n_k} \geq a_n$.

Therefore, we see that $a_n \leq b \leq b_n$ for all $n$, so $b \in I_n$ for all $n$, meaning that

$$b \in \bigcap_{i=1}^{\infty} I_n,$$

so the intersection is non-empty. Since our choice of nested intervals was arbitrary, we conclude that the Nested Interval Property is true. □

**Lemma 0.2.** *The Nested Interval Property implies the Axiom of Completeness.*

*Proof.* Let $A$ be a non-empty set of real numbers that is bounded above. Since $A$ is non-empty, there exists some number $a_1$ that is not an upper bound for $A$. Since $A$ is bounded above, there exists an upper bound $b_1$ for $A$. Then $a_1 < b_1$. Let $I_1$ be the closed interval $[a_1, b_1]$, which has length $d = b_1 - a_1 > 0$. If the midpoint of the interval, $m_1 = \frac{a_1+b_1}{2}$ is an upper bound for $A$, let $b_2 = m_1$ and let $a_2 = a_1$; otherwise, let $a_2 = m_1$ and let $b_2 = b_1$. Define $I_2 = [a_2, b_2]$. Now we can iterate this process. For any $n$, let $m_n = \frac{a_n+b_n}{2}$: if $m_n$ is an upper bound for $A$, let $b_{n+1} = m_n$ and $a_{n+1} = a_n$; otherwise, let $a_{n+1} = m_n$ and $b_{n+1} = b_n$. Either way, we define $I_{n+1} = [a_{n+1}, b_{n+1}]$.

This process yields the nested intervals $I_1 \supseteq I_2 \supseteq \dots$ such that the left endpoint of each interval is not an upper bound for $A$ and the right endpoint is. By the Nested Interval Property, $\bigcap_{n=1}^{\infty} I_n$ contains a real number $b$. I claim that $b = \sup A$. First, I want to show that $b$ is an upper bound for $A$. Suppose it were not. Then there would be some $a \in A$ such that $a > b$. Let $\epsilon_0 = a - b > 0$. Since each $b_n$ was chosen to be an upper bound for $A$, we know $a \leq b_n$ for all $n$. But then, for any $N > \frac{d}{\epsilon_0}$,

$$b_N - b \geq a - b = \epsilon_0,$$

This is clearly impossible, since $b \in I_N = [a_N, b_N]$, which has length

$$\frac{d}{2^N} < \frac{d}{N} < \epsilon_0.$$

To see that $b$ is the least upper bound, let $\epsilon > 0$. Let $N > \frac{d}{\epsilon}$. Then, since none of the $a_n$'s is an upper bound for $A$, there exists $a \in A$ such that $a_N < a \leq b_N$. Now, both $a$ and $b$ are elements of $[a_N, b_N]$, which has length $\frac{d}{2^N}$, so

$$|b - a| \leq \frac{d}{2^N} < \frac{d}{N} < \epsilon.$$

Therefore, since $|b - a| = b - a$, we see that $b - a < \epsilon$ or, equivalently, $b - \epsilon < a$. Since our choice of $\epsilon > 0$ was arbitrary, Lemma 1.3.7 implies that $b = \sup A$. □

Combining Lemmas 0.1 and 0.2, we see that the Bolzano–Weierstrass Theorem implies the Nested Interval Property and the Nested Interval Property implies the Axiom of Completeness, so we conclude that the Bolzano–Weierstrass Theorem implies the Axiom of Completeness. □

3. Define the sequence $(x_n)$ recursively by setting

$$x_1 = \sqrt{2}$$
$$x_{n+1} = \sqrt{2 + x_n} \quad \text{for all } n \in \{1, 2, 3, \dots\}$$

   (a) Show that the sequence $(x_n)$ converges.

*Proof.* I will use induction to show that $(x_n)$ is a bounded, increasing sequence; then the Monotone Convergence Sequence will imply that it converges. Specifically, I claim that, for all $n \in \{1, 2, 3, \ldots\}$,

$$\sqrt{2} \leq x_n \leq x_{n+1} \leq 2.$$

**Base Case:** Clearly, since $x_1 = \sqrt{2}$ and $x_2 = \sqrt{2 + \sqrt{2}}$,

$$\sqrt{2} \leq x_1 \leq x_2 \leq 2.$$

**Inductive Step:** Suppose $\sqrt{2} \leq x_k \leq x_{k+1} \leq 2$. Then

$$x_{k+2} = \sqrt{2 + x_{k+1}} \leq \sqrt{2 + 2} = 2.$$

Also,

$$x_{k+2} = \sqrt{2 + x_{k+1}} \geq \sqrt{2 + x_k} = x_{k+1} \geq \sqrt{2},$$

since $x_{k+1} \geq x_k$. Putting these two together, we see that

$$\sqrt{2} \leq x_{k+1} \leq x_{k+2} \leq 2.$$

Therefore, by induction, we conclude that $\sqrt{2} \leq x_n \leq x_{n+1} \leq 2$ for all $n \in \{1, 2, 3, \ldots\}$. This means that $(x_n)$ is a bounded, increasing sequence, so it converges. □

(b) Let $\lambda = \lim_{n \to \infty} x_n$. Show that $\lambda^2 - \lambda - 2 = 0$.

*Proof.* Consider the subsequence $(x_{n+1}) = (x_2, x_3, \ldots)$. This is a subsequence of a convergent sequence, so Theorem 2.5.2 implies that $\lambda = \lim x_{n+1}$. On the other hand, by the Algebraic Limit Theorem

$$\lambda^2 = (\lim x_{n+1})^2 = \lim \left(x_{n+1}^2\right) = \lim(2 + x_n) = 2 + \lim x_n = 2 + \lambda.$$

Therefore, we have that $\lambda^2 = 2 + \lambda$ or, equivalently,

$$\lambda^2 - \lambda - 2 = 0,$$

as desired. □

4. A point $x$ is called a *cluster point* of the sequence $(x_n)$ if for every $\epsilon > 0$ there are infinitely many values of $n$ with $|x_n - x| < \epsilon$.

(a) Show that $x$ is a cluster point of $(x_n)$ if and only if there is a subsequence of $(x_n)$ that converges to $x$.

*Proof.* ($\Rightarrow$) Let $x$ be a cluster point of $(x_n)$. By the definition of a cluster point, for any $k \in \{1, 2, 3, \ldots\}$, there exist infinitely many values of $n$ such that $|x_n - x| < 1/k$. Pick one and call it $n_k$. Then I claim that the subsequence $(x_{n_k})$ converges to $x$. To see this, let $\epsilon > 0$ and pick $N > \frac{1}{\epsilon}$. Then, for any $k \geq N$,

$$|x_{n_k} - x| < \frac{1}{k} \leq \frac{1}{N} < \epsilon.$$

Since our choice of $\epsilon > 0$ was arbitrary, we conclude that $(x_{n_k}) \to x$.

($\Leftarrow$) Suppose there is a subsequence of $(x_n)$ that converges to $x$. Call the subsequence $(x_{n_k})$. Let $\epsilon > 0$. Then, since $(x_{n_k}) \to x$, there exists $N \in \mathbb{N}$ such that, for all $m \geq N$,

$$|x_{n_m} - x| < \epsilon.$$

Since there are infinitely many $m \geq N$, we see that $x$ is a cluster point of $(x_n)$. $\square$

(b) Show that $(x_n)$ converges to $x$ if and only if the sequence is bounded and $x$ is its only cluster point.

*Proof.* ($\Rightarrow$) Suppose $(x_n) \to x$. Then the sequence $(x_n)$ is bounded by Theorem 2.3.2. By Theorem 2.5.2, every subsequence of $(x_n)$ also converges to $x$. Therefore, since a number is a cluster point if and only if some subsequences converges to it (by part (a)), $x$ is the only cluster point of $(x_n)$.

($\Leftarrow$) Suppose $(x_n)$ is a bounded sequence with a single cluster point $x$. Since $x$ is the only cluster point, part (a) implies that every convergent subsequence of $(x_n)$ converges to $x$. Since $(x_n)$ is bounded, HW #5 Problem 4 implies that $(x_n) \to x$. $\square$

5. Let $(x_n)$ be a sequence of real numbers such that $|x_n - x_{n+1}| \leq \frac{1}{2^n}$ for all $n \in \{1, 2, 3, \ldots\}$. Show that $(x_n)$ converges.

*Proof.* Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > 1 - \log_2 \epsilon$. Then, for any $n > m \geq N$,

$$\begin{aligned}
|x_m - x_n| &= |x_m - x_{m+1} + x_{m+1} - x_{m+2} + \ldots + x_{n-1} - x_n| \\
&\leq |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + \ldots + |x_{n-1} - x_n| \\
&\leq \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots + \frac{1}{2^{n-1}} \\
&= \frac{1}{2^m}\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-m-1}}\right).
\end{aligned}$$

The sum inside the parentheses is less than 2, so we see that

$$|x_m - x_n| < \frac{1}{2^m} \cdot 2 = \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \epsilon.$$

Therefore, since our choice of $\epsilon > 0$ was arbitrary, we see that $(x_n)$ is a Cauchy sequence and thus, by Theorem 2.6.4, converges. $\square$