Math 317 Exam #1

Due 3:00 PM Friday, October 8.

Name: ____

You may refer to your notes, your text, your homework, or the homework solutions. Do not discuss the questions with anyone except me.

There is no time limit, and the exam is due at 3:00 PM on Friday, October 8. If you plan to ask me questions about course content, please do not look at the problems until after you do so.

Please do not sign the following until the examination is completed.

I accept full responsibility under the Haverford Honor Code for my conduct on this examination.

Signed: ____

1. Suppose (x_n) is a bounded sequence with

$$\liminf x_n = a \quad \text{and} \quad \limsup x_n = b.$$

Show that there exists a subsequence of (x_n) converging to a and a subsequence converging to b.

2. In class, we used the Axiom of Completeness (via the Nested Interval Property) to prove the Bolzano–Weierstrass Theorem. For this problem, do the opposite: use the Bolzano–Weierstrass Theorem to prove the Axiom of Completeness.

[*Hint.* Do this in two steps: show that the Bolzano–Weierstrass Theorem implies the Nested Interval Property and, independently, that the Nested Interval Property implies the Axiom of Completeness.]

3. Define the sequence (x_n) recursively by setting

$$x_1 = \sqrt{2}$$
$$x_{n+1} = \sqrt{2 + x_n} \quad \text{for all } n \in \{1, 2, 3, \ldots\}$$

   (a) Show that the sequence (x_n) converges.

   (b) Let $\lambda = \lim_{n\to\infty} x_n$. Show that $\lambda^2 - \lambda - 2 = 0$.

4. A point x is called a *cluster point* of the sequence (x_n) if for every $\epsilon > 0$ there are infinitely many values of n with $|x_n - x| < \epsilon$.

   (a) Show that x is a cluster point of (x_n) if and only if there is a subsequence of (x_n) that converges to x.

   (b) Show that (x_n) converges to x if and only if the sequence is bounded and x is its only cluster point.

5. Let (x_n) be a sequence of real numbers such that $|x_n - x_{n+1}| \leq \frac{1}{2^n}$ for all $n \in \{1, 2, 3, \ldots\}$. Show that (x_n) converges.