Math 2260 Written HW #5 Solutions

1. Suppose a 15 foot chain is hanging from a winch 15 feet above the ground. The chain weighs 3 pounds per foot.

   (a) How much work is required to wind up the entire chain?

   Answer: The force required to lift an object is equal to the weight of the object. If the bottom of the chain is y feet above the ground, then the length of the chain hanging from the winch is $15 - y$. Therefore, the weight of the chain is $3(15 - y) = 45 - 3y$. In other words, the force that the winch needs to apply to the chain when the end of the chain is y feet above the ground is

   $$F(y) = 45 - 3y.$$

   Therefore, winding up the chain requires

   $$\begin{aligned} \int_0^{15} F(y)\,dy &= \int_0^{15} (45 - 3y)\,dy \\ &= \left[45y - \frac{3y^2}{2}\right]_0^{15} \\ &= \left(45(15) - \frac{3 \cdot 15^2}{2}\right) - (0 - 0) \\ &= 3 \cdot 15^2 - \frac{3 \cdot 15^2}{2} \\ &= \frac{3 \cdot 15^2}{2} \\ &= \frac{675}{2} \end{aligned}$$

   foot-pounds of work.

   (b) How much work is required to wind up the entire chain with a 100 pound load attached to it?

   Answer: As before, the weight of the chain is $45 - 3y$ when its bottom is y feet above the ground, but now there is always a 100 pound weight attached to the end of the chain, so the total weight hanging from the winch is $45 - 3y + 100 = 145 - 3y$.
   Therefore, winding up the chain-and-load assembly requires

   $$\begin{aligned} \int_0^{15} (145 - 3y)\,dy &= \left[145y - \frac{3y^2}{2}\right]_0^{15} \\ &= 145(15) - \frac{3 \cdot 15^2}{2} \\ &= 2175 - \frac{675}{2} \\ &= \frac{3675}{2} \end{aligned}$$

   foot-pounds of work.

2. Solve the differential equation
$$2xy - 3\frac{dy}{dx} = 12x.$$

Answer: The goal is to separate variables and then integrate. Therefore, I should get all the x's on the same side, so subtract $2xy$ from both sides:
$$-3\frac{dy}{dx} = 12x - 2xy.$$
Now, if I factor the right hand side, I get
$$-3\frac{dy}{dx} = 2x(6-y).$$
My goal is still to separate variables, so I'm going to divide both sides by $6-y$ and multiply by dx:
$$-3\frac{dy}{6-y} = 2x\,dx.$$
Now I can integrate:
$$\int -3\frac{dy}{6-y} = \int 2x\,dx$$
$$3\ln(|6-y|) = x^2 + C.$$
Therefore,
$$\ln(|6-y|) = \frac{x^2}{3} + C',$$
where $C' = \frac{C}{3}$ is another constant. Now, exponentiating both sides yields
$$|6-y| = e^{x^2/3+C'} = Ae^{x^2/3}$$
for the constant $A = e^{C'}$. If $6-y>0$, then $|6-y| = 6-y$, so we have
$$6 - y = Ae^{x^2/3}$$
and so
$$y = 6 - Ae^{x^2/3}.$$
On the other hand, if $6-y<0$, then $|6-y| = -(6-y) = y-6$, so we have
$$y - 6 = Ae^{x^2/3},$$
meaning that
$$y = 6 + Ae^{x^2/3}.$$
The only difference between these two expressions is in the sign of the coefficient of $e^{x^2/3}$ and so, by changing the sign of the constant A if necessary,it follows that
$$y = 6 + Ae^{x^2/3}$$
for some constant A.