Math 2260 Exam #3 Practice Problem Solutions

1. Does the following series converge or diverge? Explain your answer.

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n + n^3}.$$

Answer: Since $3^n + n^3 > 3^n$ for all $n \geq 1$, it follows that

$$\frac{2^n}{3^n + n^3} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n + n^3} < \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{2}{3}} = 3.$$

Hence, the given series converges.

2. Does the following series converge or diverge? Explain your answer.

$$\sum_{n=1}^{\infty} \frac{n}{3^n}.$$

Answer: Use the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \lim_{n\to\infty} \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} = \lim_{n\to\infty} \frac{1}{3} \cdot \frac{n+1}{n} = \frac{1}{3}.$$

Since $\frac{1}{3} < 1$, the Ratio Test implies that this series converges.

3. Does the following series converge or diverge? Explain your answer.

$$\sum_{n=1}^{\infty} 2n \sin\left(\frac{1}{n}\right).$$

Answer: Notice that the terms of this series are not going to zero:

$$\begin{aligned}
\lim_{n\to\infty} 2n \sin\left(\frac{1}{n}\right) &= \lim_{x\to\infty} 2x \sin\left(\frac{1}{x}\right) \\
&= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{2x}} \\
&= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right) \cdot \frac{-2}{x^2}}{\frac{-2}{(2x)^2}} \\
&= \lim_{x\to\infty} \frac{-2\cos\left(\frac{1}{x}\right)}{x^2} \cdot \frac{4x^2}{-2} \\
&= \lim_{x\to\infty} 4\cos\left(\frac{1}{x}\right) \\
&= 4
\end{aligned}$$

where I went from the second to the third lines using L'Hôpital's Rule. Since the limit of the terms is equal to 4, not zero, the series must diverge.

4. Does the following series converge or diverge? If it converges, find the sum. If it diverges, explain why.

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}.$$

Answer: Re-writing slightly, the given series is equal to

$$\sum_{n=1}^{\infty} \left(\frac{2^n}{4^n} + \frac{3^n}{4^n} \right) = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n}.$$

Since both of these series are convergent geometric series, I know the original series converges, so it remains only to determine the sum. Notice that

$$\sum_{n=1}^{\infty} \frac{2^n}{4^n} = \frac{2}{4} + \frac{4}{16} + \frac{8}{64} + \ldots = \frac{2}{4}\left(1 + \frac{2}{4} + \frac{4}{16} + \ldots\right) = \sum_{n=1}^{\infty} \frac{2}{4}\left(\frac{2}{4}\right)^{n-1} = \frac{2/4}{1 - 2/4} = \frac{1/2}{1/2} = 1.$$

Similarly,

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots = \frac{3}{4}\left(1 + \frac{3}{4} + \frac{9}{16} + \ldots\right) = \sum_{n=1}^{\infty} \frac{3}{4}\left(\frac{3}{4}\right)^{n-1} = \frac{3/4}{1 - 3/4} = \frac{3/4}{1/4} = 3.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n} = 1 + 3 = 4.$$

5. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2 3^n}.$$

Answer: We use the Ratio Test on the series of absolute values to first determine the radius of convergence:

$$\lim_{n\to\infty} \frac{\left|\frac{(2x-5)^{n+1}}{(n+1)^2 3^{n+1}}\right|}{\left|\frac{(2x-5)^n}{n^2 3^n}\right|} = \lim_{n\to\infty} \frac{|2x-5|^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n 3^n}{|2x-5|^n} = \lim_{n\to\infty} \frac{|2x-5|}{3} \cdot \frac{n^2}{(n+1)^2} = \frac{|2x-5|}{3}.$$

Therefore, the given series converges absolutely when $\frac{|2x-5|}{3} < 1$, meaning when $|2x-5| < 3$.

Now we check the endpoints. When $2x - 5 = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges.

Likewise, when $2x - 5 = -3$, then series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which also converges.

Therefore, the series converges for all x so that

$$-3 \leq 2x - 5 \leq 3,$$

which is the interval $[1, 4]$.

6. Use the first two non-zero terms of an appropriate Taylor series to approximate

$$\int_0^1 \sin(x^2)\,dx.$$

Estimate the error of your approximation (i.e. the difference between your answer and the actual value of the integral).

Answer: First, recall that the Taylor series centered at $x = 0$ for $\sin(x)$ is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

Therefore, the Taylor series centered at $x = 0$ for $\sin(x^2)$ is

$$\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots.$$

Hence,

$$\int_0^1 \sin(x^2)\,dx = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots\right) dx = \left[\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \dots\right]_0^1 = \frac{1}{3} - \frac{1}{42} + \frac{1}{11\cdot 5!} - \dots.$$

Therefore, we can approximate this number by

$$\frac{1}{3} - \frac{1}{42} = \frac{13}{42},$$

and we know the error is no bigger than

$$\frac{1}{11\cdot 5!} = \frac{1}{11\cdot 120} = \frac{1}{1320},$$

so in particular the estimate $\frac{13}{42}$ is accurate to 3 decimal places. (It would be totally fine in an exam situation to leave your answer as $\frac{1}{11\cdot 5!}$.)

7. Find the radius of convergence of the Taylor series

$$\sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{1+n}}{n^2}(x-2)^n.$$

Answer: Use the Ratio Test on the series of absolute values:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}\sqrt{1+(1+n)}}{(n+1)^2}(x-2)^{n+1}\right|}{\left|\frac{(-1)^n\sqrt{1+n}}{n^2}(x-2)^n\right|} &= \lim_{n\to\infty} \frac{\sqrt{2+n}\,|x-2|^{n+1}}{(n+1)^2}\cdot\frac{n^2}{\sqrt{1+n}\,|x-2|^n} \\
&= \lim_{n\to\infty} |x-2|\cdot\frac{\sqrt{2+n}}{\sqrt{1+n}}\cdot\frac{n^2}{(n+1)^2} \\
&= |x-2|.
\end{aligned}$$

Therefore, the series converges absolutely when $|x-2| < 1$, so the radius of convergence is equal to 1.

8. Write the second-degree Taylor polynomial for $f(x) = \sqrt{x}$ centered at $c = 25$. Use this polynomial to approximate $\sqrt{26}$ and estimate the error of this approximation.

Answer: I can write down the Taylor series centered at $x = 25$ for the function $f(x) = \sqrt{x}$ by first computing the derivatives of f:

$$\begin{aligned} f'(x) &= \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \\ f''(x) &= \frac{1}{2} \cdot \frac{-1}{2}x^{-3/2} = \frac{-1}{4x^{3/2}} \\ f'''(x) &= \frac{-1}{4} \cdot \frac{-3}{2}x^{-5/2} = \frac{3}{8x^{5/2}} \end{aligned}$$

Therefore,

$$\begin{aligned} f(25) &= \sqrt{25} = 5 \\ f'(25) &= \frac{1}{2\sqrt{25}} = \frac{1}{10} \\ f''(25) &= \frac{-1}{4 \cdot 25^{3/2}} = \frac{-1}{4 \cdot 125} = \frac{-1}{500} \\ f'''(25) &= \frac{3}{8 \cdot 25^{5/2}} = \frac{3}{8 \cdot 3125} = \frac{3}{25,000} \end{aligned}$$

Hence, the Taylor series centered at $x = 25$ for $\sqrt{x}$ is

$$\sqrt{x} = 5 + \frac{(x-25)}{10} - \frac{(x-25)^2}{1000} + \frac{3(x-25)^3}{150,000} - \dots$$

Therefore,

$$\sqrt{26} \approx 5 + \frac{(26-25)}{10} - \frac{(26-25)^2}{1000} = 5 + \frac{1}{10} - \frac{1}{1000} = 5.1 - 0.001 = 5.099,$$

with an error no more than

$$\frac{3(26-25)^3}{150,000} = \frac{3}{150,000} = \frac{1}{50,000} = 0.00002.$$

(In fact, $\sqrt{26} \approx 5.0990195$.)

9. Does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$$

converge or diverge. Be sure to give a complete explanation.

Answer: Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$ as we discussed in class, a limit comparison to the series $\sum \frac{1}{n^2}$ is a natural:

$$\lim_{n\to\infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Therefore, since the series $\sum \frac{1}{n^2}$ converges, the Limit Comparison Test implies that the given series converges as well.

10. Does the following series converge or diverge? Explain your answer.

$$\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}.$$

Answer: Use the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{(n+1)!(n+2)!}{(3(n+1))!}}{\frac{n!(n+1)!}{(3n)!}} = \lim_{n\to\infty} \frac{(n+1)!(n+2)!}{(3n+3)!} \cdot \frac{(3n)!}{n!(n+1)!} = \lim_{n\to\infty} \frac{(n+1)(n+2)}{(3n+3)(3n+2)(3n+1)} = 0,$$

since the numerator is a polynomial of degree 2 but the denominator is a polynomial of degree 3.

Therefore, since $0 < 1$ the Ratio Test implies that the series converges.

11. Does the sequence

$$\left(\arctan\left(\frac{n^2}{n^2+1}\right)\right)_{n=1}^{\infty}$$

converge or diverge? If it converges, find the limit; if it diverges, explain why.

Answer: First, notice that

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Therefore, the term inside the arctangent is going to 1, so

$$\lim_{n\to\infty} \arctan\left(\frac{n^2}{n^2+1}\right) = \arctan(1) = \frac{\pi}{4}.$$

12. Does the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-\sqrt{n}}$$

converge or diverge? Explain your answer.

Answer: For large n, the n^2 should dominate the $\sqrt{n}$, so let's do a limit comparison to the convergent series $\sum \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-\sqrt{n}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{n^2-\sqrt{n}} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2}{n^2-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^{3/2}}} = 1.$$

Therefore, since $\sum \frac{1}{n^2}$ converges, so does $\sum \frac{1}{n^2-\sqrt{n}}$.

13. Does the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^5}$$

converge or diverge? Explain your answer.

Answer: Use the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^5}}{\frac{n!}{n^5}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^5} \cdot \frac{n^5}{n!} = \lim_{n\to\infty} (n+1)\frac{n^5}{(n+1)^5} = \infty$$

since the expression $\frac{n^5}{(n+1)^5}$ is going to 1 but $(n+1)$ is going to ∞.

Therefore, the Ratio Test implies that the series diverges.

14. Does the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^3}$$

converge or diverge? Explain your answer.

Answer: Use the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)^3}}{\frac{3^n}{n^3}} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n} = \lim_{n\to\infty} 3 \cdot \frac{(n+1)^3}{n^3} = 3.$$

Since $3 > 1$, the Ratio Test implies that the series diverges.

15. Does the series

$$\sum_{n=0}^{\infty} \frac{2n+3}{(n^2+3n+6)^2}$$

converge or diverge? Explain your answer.

Answer: For n very large, the denominator will be dominated by the term n^4, so do a limit comparison to the convergent series $\sum \frac{n}{n^4}$:

$$\lim_{n\to\infty} \frac{\frac{2n+3}{(n^2+3n+6)^2}}{\frac{n}{n^4}} = \lim_{n\to\infty} \frac{2n+3}{(n^2+3n+6)^2} \cdot \frac{n^4}{n} = \lim_{n\to\infty} \frac{2n+3}{n} \cdot \frac{n^4}{(n^2+3n+6)^2} = 2 \cdot 1 = 2.$$

Therefore, since the limit is finite and the series $\sum \frac{n}{n^4} = \frac{1}{n^3}$ converges, the Limit Comparison Test implies that the given series converges as well.

16. For which values of x does the series

$$\sum_{n=0}^{\infty} \frac{(x-4)^n}{5^n}$$

converge? What is the sum of the series when it converges?

Answer: First, use the Ratio Test on the series of absolute values:

$$\lim_{n\to\infty} \frac{\left|\frac{(x-4)^{n+1}}{5^{n+1}}\right|}{\left|\frac{(x-4)^n}{5^n}\right|} = \lim_{n\to\infty} \frac{|x-4|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{|x-4|^n} = \frac{|x-4|}{5},$$

so the given series converges absolutely whenever $\frac{|x-4|}{5} < 1$, meaning when $|x-4| < 5$ (from this we see that the radius of convergence of the series is 5).

Now check the endpoints. When $x - 4 = 5$, the series becomes

$$\sum_{n=0}^{\infty} \frac{5^n}{5^n} = \sum_{n=0}^{\infty} 1,$$

which diverges.

Similarly, when $x - 4 = -5$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-5)^n}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n,$$

which also diverges.

Therefore, the series converges for

$$-5 < x - 4 < 5,$$

which is to say, on the interval $(-1, 9)$.

When the series does converge, it is just the geometric series

$$\sum_{n=0} \left(\frac{x-4}{5}\right)^n = \frac{1}{1 - \frac{x-4}{5}} = \frac{1}{\frac{5-(x-4)}{5}} = \frac{5}{9-x},$$

so when it converges the series converges to the function $f(x) = \frac{5}{9-x}$.

17. Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+n^3)}{n^4+1}$$

converge absolutely, converge conditionally, or diverge? Explain your answer.

Answer: The series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n(n^2+n^3)}{n^4+1} \right| = \sum_{n=1}^{\infty} \frac{n^2+n^3}{n^4+1}$$

should behave similarly to $\sum \frac{n^3}{n^4}$, so do a limit comparison to this series:

$$\lim_{n\to\infty} \frac{\frac{n^2+n^3}{n^4}}{\frac{n^3}{n^4}} = \lim_{n\to\infty} \frac{n^2+n^3}{n^4} \cdot \frac{n^4}{n^3} = \lim_{n\to\infty} \frac{n^2+n^3}{n^3} = 1.$$

Therefore, since $\sum \frac{n^3}{n^4} = \sum \frac{1}{n}$ diverges, the Limit Comparison Test says that the series of absolute values diverges as well.

However, the given series satisfies the hypotheses of the Alternating Series Test, so it converges. Therefore, since the series converges but the series of absolute values diverges, we conclude that the series converges conditionally.

18. Does the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^n(n!)^2}$$

converge or diverge? Explain your answer.

Answer: Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{(2(n+1))!}{2^{n+1}((n+1)!)^2}}{\frac{(2n)!}{2^n(n!)^2}} &= \lim_{n\to\infty} \frac{(2n+2)!}{2^{n+1}((n+1)!)^2} \cdot \frac{2^n(n!)^1}{(2n)!} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2} \cdot \frac{(n!)^2}{((n+1)n!)^2} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2} \cdot \frac{1}{(n+1)^2}.
\end{aligned}$$

Since $(2n+2)(2n+1) = 4n^2+6n+2$ and since $(n+1)^2 = n^2+2n+1$, the above limit is equal to

$$\lim_{n\to\infty} \frac{4n^2+6n+2}{2(n^2+2n+1)} = \frac{4}{2} = 2.$$

Since $2 > 1$, the Ratio Test implies that the given series diverges.

19. Does the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$$

converge absolutely, converge conditionally, or diverge? Explain your answer.

Answer: The series of absolute values

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n\ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$$

diverges (see HW #12, Problem 3 for a proof of this). However, the series satisfies the hypotheses of the Alternating Series Test and hence converges, so we see that it converges conditionally.

20. For what values of p does the series

$$\sum_{n=0}^{\infty} \frac{1}{(n^2+1)^p}$$

converge? Explain your answer.

Answer: For large n, the $+1$ will barely contribute, so do a limit comparison with the series $\sum \frac{1}{(n^2)^p}$:

$$\lim_{n\to\infty} \frac{\frac{1}{(n^2+1)^p}}{\frac{1}{(n^2)^p}} = \lim_{n\to\infty} \frac{1}{(n^2+1)^p} \cdot \frac{(n^2)^p}{1} = \lim_{n\to\infty} \left(\frac{n^2}{n^2+1}\right)^p = 1^p = 1,$$

since $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$.

Therefore, the given series and the series $\sum \frac{1}{(n^2)^p} = \sum \frac{1}{n^{2p}}$ will either both converge or both diverge.

Since $\sum \frac{1}{n^{2p}}$ converges for $2p > 1$ and diverges otherwise, we see that the given series converges when $2p > 1$, which is to say when

$$p > \frac{1}{2}.$$

21. What is the interval of convergence of the following power series? Explain your answer.

$$\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n^2}.$$

Answer: Start by applying the Ratio Test to the series of absolute values:

$$\lim_{n\to\infty} \frac{\left|\frac{3^{n+1}(x-2)^{n+1}}{(n+1)^2}\right|}{\left|\frac{3^n(x-2)^n}{n^2}\right|} = \lim_{n\to\infty} \frac{3^{n+1}|x-2|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{3^n|x-2|^n} = \lim_{n\to\infty} 3|x-2| \cdot \frac{n^2}{(n+1)^2} = 3|x-2|.$$

Therefore, the ratio test implies that the given series converges absolutely when $3|x-2| < 1$, meaning when $|x-2| < \frac{1}{3}$.

Now, check the endpoints. When $x - 2 = \frac{1}{3}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n\left(\frac{1}{3}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges.

Likewise, when $x-2=-\frac{1}{3}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n\left(-\frac{1}{3}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which also converges.

Therefore, the series converges for

$$-\frac{1}{3} \le x-2 \le \frac{1}{3},$$

meaning the interval of convergence is $\left[\frac{5}{3}, \frac{7}{3}\right]$.

22. What are the first four nonzero terms of the Taylor series centered at $x=0$ for the function $f(x)=xe^{3x}$?

Answer: Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$

we see that

$$e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots = 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \ldots.$$

Therefore,

$$xe^{3x} = x\left(1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \ldots\right) = x + 3x^2 + \frac{9x^3}{2!} + \frac{27x^4}{3!} + \ldots.$$

23. Which of the following gives the value of $\int_0^{1/2} \cos(x^2)\,dx$ correct to within 0.0001 (i.e. within $1/10,000$)? Explain your answer.

(a) $\frac{1}{2}$ (b) $\frac{1}{2} - \frac{1}{120}$ (c) $\frac{1}{24} - \frac{1}{2688}$ (d) $\frac{1}{2} - \frac{1}{320}$ (e) $\frac{1}{2} - \frac{1}{384}$ (f) $\frac{1}{24} - \frac{1}{1024}$

Answer: Since

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots,$$

the Taylor series for $\cos(x^2)$ centered at $x=0$ is

$$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \ldots = 1 - \frac{x^4}{2} + \frac{x^8}{24} - \ldots.$$

Therefore,

$$\begin{aligned}\int_0^{1/2} \cos(x^2)\,dx &= \int_0^{1/2} \left(1 - \frac{x^4}{2} + \frac{x^8}{24} - \ldots\right) dx \\ &= \left[x - \frac{x^5}{10} + \frac{x^9}{9\cdot 24} - \ldots\right]_0^{1/2} \\ &= \frac{1}{2} - \frac{1}{320} + \frac{1}{9\cdot 24\cdot 2^9} - \ldots.\end{aligned}$$

Hence, choice (d) is clearly the best approximation.

24. What is the limit of the following sequence? Explain your answer.

$$\Big(\sin(\arctan(\ln(n)))\Big)_{n=2}^{\infty}$$

Answer: Since $\lim_{n\to\infty} \ln(n) = +\infty$ and since $\lim_{x\to\pi/2^+} \tan(x) = +\infty$, it follows that

$$\lim_{n\to\infty} \arctan(\ln(n)) = \frac{\pi}{2}.$$

Therefore,

$$\lim_{n\to\infty} \sin(\arctan(\ln(n))) = \sin(\pi/2) = 1.$$

25. Does the series

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2 - n + 1}$$

converge or diverge? Explain your answer.

Answer: For large n, we should expect that n^2 will dominate the denominator, so do a limit comparison to $\sum \frac{\sqrt{n}}{n^2}$:

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2-n+1}}{\frac{\sqrt{n}}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2 - n + 1} \cdot \frac{n^2}{\sqrt{n}} = \lim_{n\to\infty} \frac{n^2}{n^2 - n + 1} = 1.$$

Therefore, since the series $\sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$ converges, so does the given series.

26. Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n^2 + n^3)}{n^4 + \ln(n)}$$

converge absolutely, converge conditionally, or diverge? Explain your answer.

Answer: First, notice that the series satisfies the Hypotheses of the Alternating Series Test, so it definitely converges. To see whether it converges absolutely, consider the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n (n^2 + n^3)}{n^4 + \ln(n)} \right| = \sum_{n=1}^{\infty} \frac{n^2 + n^3}{n^4 + \ln(n)}.$$

Now, you should expect the n^3 to dominate the numerator and the n^4 to dominate the denominator, so do a limit comparison to the series $\sum \frac{n^3}{n^4}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{n^2+n^3}{n^4+\ln(n)}}{\frac{n^3}{n^4}} &= \lim_{n\to\infty} \frac{n^2 + n^3}{n^4 + \ln(n)} \cdot \frac{n^4}{n^3} \\
&= \lim_{n\to\infty} \frac{n^2 + n^3}{n^3} \cdot \frac{n^4}{n^4 + \ln(n)} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n} + 1}{1} \cdot \frac{1}{1 + \frac{\ln(n)}{n^4}} \\
&= 1 \cdot 1 \\
&= 1.
\end{aligned}$$

Therefore, since the series $\sum \frac{n^3}{n^4} = \sum \frac{1}{n}$ diverges, the Limit Comparison Test implies that the series of absolute values diverges, and hence the given series only converges conditionally.

27. If it converges, find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n}.$$

If the series diverges, explain why.

Answer: This is tricky. Remember that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

But now if I let $x = -1/2$ on the right hand side, I get the series

$$\sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n},$$

which is the given series. Hence, the given series sums to

$$e^{-1/2} = \frac{1}{\sqrt{e}}.$$

28. For which values of p does the following series converge? Explain your answer.

$$\sum_{n=1}^{\infty} \sqrt{1+n^p}.$$

Answer: Notice that for any p, $1 + n^p > 1$, meaning that

$$\sqrt{1+n^p} > \sqrt{1} = 1.$$

Therefore,

$$\sum_{n=1}^{\infty} \sqrt{1+n^p} > \sum_{n=1}^{\infty} 1,$$

which obviously diverges, so the given series also diverges.

29. What is the interval of convergence of the following series? Explain your answer.

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}.$$

Answer: First, use the Ratio Test on the series of absolute values:

$$\lim_{n\to\infty} \frac{\left|\frac{(x-1)^{n+1}}{(n+1)^2}\right|}{\left|\frac{(x-1)^n}{n^2}\right|} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|x-1|^n} = \lim_{n\to\infty} |x-1| \cdot \frac{n^2}{(n+1)^2} = |x-1|.$$

Thus, the Ratio Test says that the given series converges absolutely when $|x-1| < 1$.

Now check the endpoints. When $x - 1 = 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges.

Likewise, when $x - 1 = -1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which also converges.

Therefore, the given series converges for

$$-1 \leq x - 1 \leq 1,$$

so the interval of convergence is $[0, 2]$.