Math 113 Exam #1 Solutions

1. What are the domain and range of the function $f(x) = \frac{e^x}{\sqrt{x^4-1}}$?

   Answer: Notice that $\sqrt{x^4 - 1}$ is only defined when $x^4 - 1$ is non-negative, meaning that $x \leq -1$ or $x \geq 1$. Moreover, the function f is only defined when the denominator is non-zero, which excludes $x = \pm 1$. Therefore, the domain of f is

   $$\{x : x < -1 \text{ or } x > 1\} = (-\infty, -1) \cup (1, +\infty).$$

   As for the range, notice that both the numerator and the denominator are always positive, so negative numbers and zero are definitely not in the range. As x gets very negative, the numerator gets very close to zero. Also, as x gets close to -1, the denominator gets very close to zero while the numerator goes to $1/e > 0$. Hence, the range of f consists of all positive numbers:

   $$\text{Range}(f) = \{y : 0 < y\} = (0, +\infty).$$

2. Let $f(x) = e^{3x-2}$. Is f invertible? Why or why not? If f is invertible, what is $f^{-1}(x)$?

   Answer: Yes, f is invertible. To see why, notice, first of all, that $3x - 2$ is an increasing function and passes the horizontal line test. Moreover, e^x is also an increasing function that passes the horizontal line test. Hence, the composition $f(x) = e^{3x-2}$ is also an increasing function, so it has an inverse.

   We can find the inverse by swapping x and y in the expression $y = e^{3x-2}$ and solving for y. Swapping x and y yields

   $$x = e^{3y-2}.$$

   Taking the natural logarithm of both sides, we see that

   $$\ln x = \ln\left(e^{3y-2}\right) = 3y - 2.$$

   Adding 2 to both sides gives

   $$\ln x + 2 = 3y.$$

   Now, just divide both sides by 3:

   $$\frac{\ln x + 2}{3} = y.$$

   Therefore, the inverse of f is given by

   $$f^{-1}(x) = \frac{\ln x + 2}{3}.$$

3. For each of the following, either evaluate the limit or explain why it doesn't exist.

   (a)

   $$\lim_{x \to 16} \frac{4 - \sqrt{x}}{x - 16}$$

   Answer: I'm going to consider $h(x) = \frac{4-\sqrt{x}}{x-16}$. Rationalizing the denominator yields

   $$\frac{4 - \sqrt{x}}{x - 16} \cdot \frac{4 + \sqrt{x}}{4 + \sqrt{x}} = \frac{16 - x}{(x - 16)(4 + \sqrt{x})} = \frac{-(x - 16)}{(x - 16)(4 + \sqrt{x})}.$$

   As long as $x \neq 16$, this is equal to

   $$g(x) = \frac{-1}{4 + \sqrt{x}}.$$

Therefore, since $h(x)$ and $g(x)$ agree except when $x = 16$, we know that

$$\lim_{x\to 16} h(x) = \lim_{x\to 16} g(x) = \lim_{x\to 16} \frac{-1}{4+\sqrt{x}}.$$

Since $g(x)$ is a quotient of algebraic functions, it is continuous everywhere it is defined. In particular, $g(x)$ is continuous at $x = 16$, so we can evaluate the above limit by plugging in $x = 16$:

$$\lim_{x\to 16} \frac{-1}{4+\sqrt{x}} = \frac{-1}{4+\sqrt{16}} = -\frac{1}{8}.$$

Putting this all together, we can conclude that

$$\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} = -\frac{1}{8}.$$

(b)

$$\lim_{x\to -\infty} \frac{6x^2}{\sqrt{7x^4+9}}$$

Answer: To evaluate this limit, I want to multiply numerator and denominator by $\frac{1}{x^2}$:

$$\lim_{x\to -\infty} \frac{\frac{1}{x^2}\left(6x^2\right)}{\frac{1}{x^2}\sqrt{7x^4+9}} = \lim_{x\to -\infty} \frac{6}{\sqrt{\frac{1}{x^4}\left(7x^4+9\right)}} = \lim_{x\to -\infty} \frac{6}{\sqrt{7+\frac{9}{x^4}}}.$$

Since $\frac{9}{x^4}$ goes to zero as $x \to -\infty$, we see that

$$\lim_{x\to -\infty} \frac{6}{\sqrt{7+\frac{9}{x^4}}} = \frac{6}{\sqrt{7}}.$$

Therefore, we can conclude that

$$\lim_{x\to -\infty} \frac{6x^2}{\sqrt{7x^4+9}} = \frac{6}{\sqrt{7}}.$$

(c)

$$\lim_{x\to -2} \frac{x^2+x-2}{x^2+4x+4}$$

Answer: I claim that this limit does not exist. To see why, notice, first of all, that

$$\frac{x^2+x-2}{x^2+4x+4} = \frac{(x+2)(x-1)}{(x+2)(x+2)} = \frac{x-1}{x+2}$$

so long as $x \neq -2$. Therefore, if it exists, $\lim_{x\to -2} \frac{x^2+x-2}{x^2+4x+4}$ must be equal to

$$\lim_{x\to -2} \frac{x-1}{x+2}.$$

Notice that, as $x \to -2$, the numerator goes to -3, while the denominator goes to zero. However, the sign of the denominator depends on which direction x approaches -2 from. When x approaches -2 from the left, we have that

$$\lim_{x\to -2^-} \frac{x-1}{x+2} = +\infty,$$

since both the numerator and denominator are negative. However, as x approaches -2 from the right, we have that

$$\lim_{x \to -2^+} \frac{x-1}{x+2} = -\infty,$$

since the numerator is negative and the denominator is positive.

Therefore, since the two one-sided limits do not agree, the given limit does not exist.

4. (a) At which numbers is the function $h(x) = \cos\left(\frac{x}{1-x^2}\right)$ continuous? Justify your answer.

   Answer: I claim that $h(x)$ is continuous whenever $x \neq \pm 1$. To see this, notice that $g(x) = \frac{x}{1-x^2}$ is a rational function, so it is continuous wherever it is defined. Since this function is defined so long as $1 - x^2 \neq 0$, we see that it is defined for all $x \neq \pm 1$.

   In turn, the function $f(x) = \cos x$ is continuous, and we know that the composition of continuous functions is continuous. Hence,

$$h(x) = (f \circ g)(x)$$

   is continuous wherever it is defined, namely for all $x \neq \pm 1$.

   (b) What is $\lim_{x \to 0} h(x)$? Explain your reasoning.

   Answer: From part (a), we know that $h(x)$ is continuous at $x = 0$. Therefore, by definition of continuity,

$$\lim_{x \to 0} h(x) = h(0) = \cos\left(\frac{0}{1-0^2}\right) = \cos(0) = 1.$$

5. Suppose, for some bizarre reason, NASA built a giant landing platform at the level of the cloud tops on Jupiter. If you're standing on the platform and toss a ball straight up with an initial velocity of 32 m/s, its height (in meters) above the platform is given (approximately) by

$$s(t) = -13t^2 + 32t.$$

Given that, what is the velocity of the ball 2 seconds after you release it?

Answer: We know that the velocity of the ball at time $t = 2$ is given by $s'(2)$. Therefore, by the definition of the derivative, the velocity is given by

$$\begin{aligned} s'(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} &= \lim_{h \to 0} \frac{\left[-13(2+h)^2 + 32(2+h)\right] - \left[-13(2)^2 + 32(2)\right]}{h} \\ &= \lim_{h \to 0} \frac{\left[-13(4+4h+h^2) + 32 \cdot 2 + 32h\right] - \left[-13 \cdot 4 + 32 \cdot 2\right]}{h} \\ &= \lim_{h \to 0} \frac{-13 \cdot 4 - 13 \cdot 4h - 13h^2 + 32 \cdot 2 + 32h + 13 \cdot 4 - 32 \cdot 2}{h} \end{aligned}$$

We can cancel the factors $-13 \cdot 4$ and $32 \cdot 2$, yielding:

$$\lim_{h \to 0} \frac{-13 \cdot 4h - 13h^2 + 32h}{h} = \lim_{h \to 0} (-13 \cdot 4 - 13h + 32) = -13 \cdot 4 + 32.$$

Some quick arithmetic gives us that $-13 \cdot 4 = -52$, so the above is equal to -20.

Therefore, we can conclude that the velocity of the ball 2 seconds after you release it is -20 m/s, so it is falling at a rate of 20 m/s.