Fall 2009 Math 113 Exam #2 Solutions

1. An asteroid hits the Atlantic Ocean and creates an expanding circular wave. If the area enclosed by this wave increases at the rate of 200 km^2/min, how fast is the *diameter* of the wave expanding when its *radius* is 20 km?

   Answer: We know that the area enclosed by the wave is given by $A(t) = \pi r(t)^2$ and that the diameter is $2r(t)$. From this latter, the rate of change of the diameter (which is what we're trying to determine) is $2r'(t)$.

   Differentiating the expression for A,
$$A'(t) = 2\pi r(t)r'(t).$$
   Therefore,
$$r'(t) = \frac{A'(t)}{2\pi r(t)}.$$
   At the time t_0 that we're interested in, $A'(t_0) = 200$ and $r(t_0) = 20$. Plugging these values into the above equation, we see that
$$r'(t_0) = \frac{200}{2\pi(20)} = \frac{200}{40\pi} = \frac{5}{\pi}.$$
   Therefore, at this instant the diameter is increasing at a rate of
$$2r'(t_0) = 2\frac{5}{\pi} = \frac{10}{\pi} \text{ km/min}.$$

2. Suppose 10 bacteria are left to grow in a petri dish. At least for the first few hours, the rate of population growth of this bacterial culture will be proportional to the size of the culture. If there are 80 bacteria after 1 hour, how many will there be after 100 minutes?

   Answer: Since the rate of growth is proportional to the size of the population, the population function is given by
$$P(t) = Ce^{kt}$$
   for some constants C and k. To determine C, we plug in $t = 0$:
$$10 = P(0) = Ce^{k\cdot 0} = C,$$
   so the population function is
$$P(t) = 10e^{kt}.$$
   To determine k, we plug in $t = 1$:
$$80 = P(1) = 10e^{k\cdot 1} = 10e^k.$$
   Hence,
$$e^k = 8$$
   and, taking the natural log of both sides,
$$k = \ln 8.$$
   Thus, the population function is
$$P(t) = 10e^{\ln 8t} = 10\left(e^{\ln 8}\right)^t = 10(8)^t.$$
   Therefore, since 100 minutes $= \frac{5}{3}$ hours, the population after 100 minutes is
$$P(5/3) = 10(8)^{5/3} = 10 \cdot 32 = 320.$$

3. Find the equation of the tangent line to the graph of $2e^{xy} = x + y$ at the point $(0, 2)$.

   Answer: We use implicit differentiation to take the derivative of both sides with respect to x:

$$2e^{xy}\left(x\frac{dy}{dx} + y\right) = 1 + \frac{dy}{dx}.$$

   Therefore,

$$(2xe^{xy} - 1)\frac{dy}{dx} = 1 - 2ye^{xy}$$

   or, equivalently,

$$\frac{dy}{dx} = \frac{1 - 2ye^{xy}}{2xe^{xy} - 1}.$$

   Therefore, the slope of the tangent line at the point $(0, 2)$ is

$$\frac{1 - 2(2)e^{0\cdot 2}}{2(0)e^{0\cdot 2} - 1} = \frac{1 - 4}{-1} = 3.$$

   Hence, using the point-slope formula, the equation of the tangent line is

$$y - 2 = 3(x - 0),$$

   or, equivalently,

$$y = 3x + 2$$

4. At what x values does the graph of the function $f(x) = \tan(1 - x^2)$ have a horizontal tangent line?

   Answer: The graph will have a horizontal tangent at all values of x for which $f'(x) = 0$. Now,

$$f'(x) = \sec^2(1 - x^2) \cdot (-2x) = -2x\sec^2(1 - x^2).$$

   Since

$$\sec^2\theta = \frac{1}{\cos^2\theta}$$

   is never zero for any value of θ, we see that $f'(x) = 0$ only if $x = 0$, so the graph of $f(x)$ has a horizontal tangent line only at $x = 0$.

5. Let $y = x^{\ln x}$. What is $\frac{dy}{dx}$?

   Answer: First, take the natural log of both sides:

$$\ln y = \ln\left(x^{\ln x}\right) = \ln x \ln x.$$

   Now, differentiating both sides with respect to x, we see that

$$\frac{1}{y}\frac{dy}{dx} = \ln x\frac{1}{x} + \frac{1}{x}\ln x = 2\frac{\ln x}{x}.$$

   Therefore,

$$\frac{dy}{dx} = 2y\frac{\ln x}{x}.$$

   Plugging in $y = x^{\ln x}$, we see that

$$\frac{dy}{dx} = 2x^{\ln x}\frac{\ln x}{x}.$$

6. Estimate $(0.99)^9$ using an appropriate linearization.

Answer: Let $f(x) = x^9$. Then we will approximate $(0.99)^9$ using the linearization of f at 1. To find this linearization, we need to evaluate $f(1)$ and $f'(1)$. To that end, note that

$$f'(x) = 9x^8.$$

Hence

$$f(1) = 1^9 = 1, \qquad f'(1) = 9(1)^8 = 9.$$

Therefore, the linearization of f at 1 is

$$L(x) = f(1) + f'(1)(x-1) = 1 + 9(x-1).$$

Therefore,

$$L(0.99) = 1 + 9(0.99 - 1) = 1 + 9(-0.01) = 1 - 0.09 = 0.91,$$

so $(0.99)^9 \approx 0.91$.