

### Math 113 Exam #1 Solutions

1. What are the domain and range of the function  $f(x) = \frac{1}{\sqrt{4-x}}$ ?

**Answer:**  $f(x)$  is well-defined provided  $\sqrt{4-x} \neq 0$ . In order for  $\sqrt{4-x}$  to exist, we must have

$$4 - x \geq 0,$$

meaning that  $x \leq 4$ . In order for  $\sqrt{4-x} \neq 0$  it must also be the case that  $x \neq 4$ , so the domain of  $f$  is

$$(-\infty, 4).$$

Since  $\sqrt{4-x} > 0$  where  $f$  is defined, we see that  $f(x) > 0$  for all  $x$  in the domain, so the range of  $f$  is

$$(0, +\infty).$$

2. Evaluate

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 - 8x + 7}}{17x + 12}$$

**Answer:** Dividing both numerator and denominator by  $x$  yields

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}\sqrt{4x^2 - 8x + 7}}{\frac{1}{x}(17x + 12)} &= \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{1}{x^2}(4x^2 - 8x + 7)}}{17 + \frac{12}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{4 - \frac{8}{x} + \frac{7}{x^2}}}{17 + \frac{12}{x}} \\ &= \frac{\sqrt{4}}{17} \\ &= \frac{2}{17}. \end{aligned}$$

3. Let

$$f(x) = \begin{cases} \frac{x-2}{x^2-4} & \text{for } x \neq 2 \\ a & \text{for } x = 2 \end{cases}$$

If  $f(x)$  is continuous at  $x = 2$ , then find the value of  $a$ .

**Answer:** In order for  $f$  to be continuous at  $x = 2$ , we must have that

$$a = f(2) = \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x-2}{x^2-4}.$$

Now, we can factor the denominator in the limit to get

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x+2)(x-2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4},$$

so we see that, in order for  $f$  to be continuous,  $a$  must be  $\frac{1}{4}$ .

4. Are there any solutions to the equation  $\cos x = x$ ?

**Answer:** Yes, there is a solution to the equation. To see this, let

$$f(x) = \cos x - x.$$

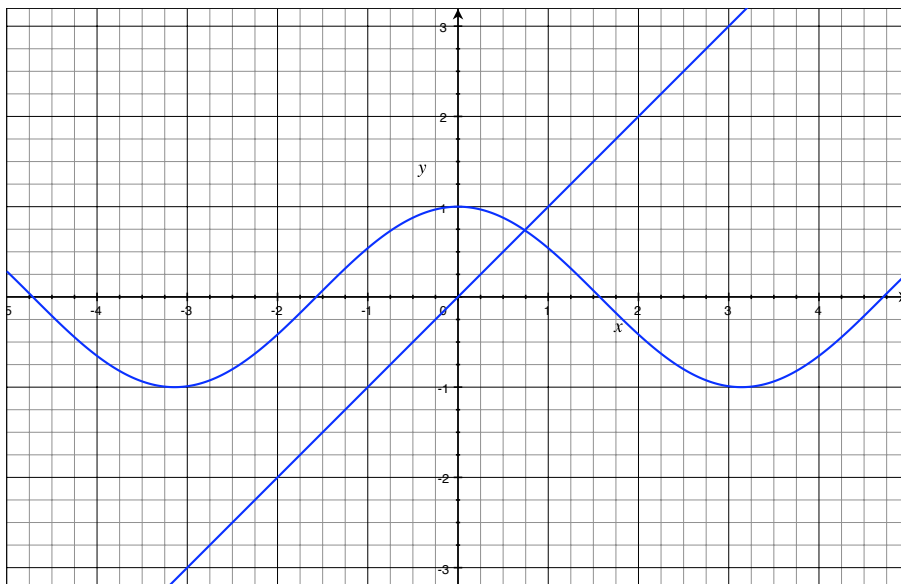
Then  $f(x) = 0$  precisely when  $x$  is a solution to the equation  $\cos x = x$ , so the problem is to show that  $f(x) = 0$  for some  $x$ . To see this, notice that  $f$  is continuous (since both  $\cos x$  and  $x$  are continuous functions) and

$$f(0) = \cos 0 - 0 = 1 > 0$$

$$f(\pi) = \cos \pi - \pi = -1 - \pi < 0.$$

Therefore, by the Intermediate Value Theorem there exists  $c$  between 0 and  $\pi$  such that  $f(c) = 0$ . Then, as noted above,  $\cos c = c$ , so  $c$  is a solution to the equation.

Intuitively, we can see that  $\cos x = x$  has a solution by looking at the graphs of  $y = \cos x$  and  $y = x$ ; they intersect in exactly one point, so the solution  $c$  that we saw exists actually be the only solution.



5. Determine the following limits, if they exist

(a)  $\lim_{x \rightarrow -1} \frac{x^2 - 2x + 1}{x - 1}$

**Answer:** Plugging in  $x = -1$  yields that

$$\lim_{x \rightarrow -1} \frac{x^2 - 2x + 1}{x - 1} = \frac{(-1)^2 - 2(-1) + 1}{(-1) - 1} = \frac{4}{-2} = -2.$$

(b)  $\lim_{x \rightarrow 1^-} \frac{x^2 + 2x + 1}{x - 1}$

**Answer:** Notice that

$$\lim_{x \rightarrow 1} (x^2 + 2x + 1) = 1 + 2 + 1 = 4,$$

so the numerator is going to 4. Also, the denominator is going to zero, so we expect the limit to be  $\pm\infty$ . To see which, notice that, if  $x < 1$ , then

$$x - 1 < 0,$$

so the denominator is a very small negative number as  $x \rightarrow 1^-$ . Hence,

$$\lim_{x \rightarrow 1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty.$$

6. Let

$$g(x) = \sqrt{x}$$

Is  $g$  differentiable at 0? If so, what is  $g'(0)$ ?

**Answer:** If  $g$  is differentiable at 0, then, by definition,

$$\begin{aligned} g'(0) &= \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{0+h} - \sqrt{0}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}}, \end{aligned}$$

which does not exist (when  $h > 0$ , this goes to  $+\infty$ ; when  $h < 0$ , the expression  $\sqrt{h}$  doesn't even make sense in the real numbers). Therefore,  $g$  is *not* differentiable at 0.

7. Let

$$f(x) = 2x^2 + 3x.$$

Is  $f$  differentiable at 1? If so, what is  $f'(1)$ ?

**Answer:** If  $f$  is differentiable at 1, then, by definition,

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{[2(1+h)^2 + 3(1+h)] - [2(1)^2 + 3(1)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[2(1+2h+h^2) + 3(1) + 3h] - [2(1)^2 + 3(1)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{7h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (7 + h) \\ &= 7, \end{aligned}$$

so we see that  $f$  is differentiable at 1 and that  $f'(1) = 7$ .