Math 215 HW #9 Solutions

1. Problem 4.4.12. If A is a 5 by 5 matrix with all $|a_{ij}| \leq 1$, then $\det A \leq$ ____. Volumes or the big formula or pivots should give some upper bound on the determinant.

   Answer: Let $\vec{v}_i$ be the ith column of A. Then

   $$|\vec{v}_i| = \sqrt{a_{1i}^2 + a_{2i}^2 + a_{3i}^2 + a_{4i}^2 + a_{5i}^2}.$$

   Since $|a_{ij}| \leq 1$ for each i and j, it's also true that each $a_{ij}^2 \leq 1$. Hence, the right hand side of the above equation is no bigger than $\sqrt{1+1+1+1+1} = \sqrt{5}$, and so we see that

   $$|\vec{v}_i| \leq \sqrt{5}.$$

   This means that each edge of the 5-dimensional box spanned by the columns of A is no longer than $\sqrt{5}$, meaning that the volume of that box can be no bigger than

   $$\left(\sqrt{5}\right)^5 = 5^{5/2} = 25\sqrt{5}.$$

   Since $|\det A|$ is exactly equal to the volume of the box spanned by the columns of A, this reasoning implies that

   $$|\det A| \leq 25\sqrt{5} \approx 55.9.$$

   Another way to get an upper bound is to use the big formula for the determinant from p. 212. Notice that, for any 5 by 5 matrix, there are $5! = 120$ terms in the sum. When all entries of A are smaller than 1 (in absolute value), it must be the case that each of the 120 terms in the sum is smaller than 1 (in absolute value). Therefore, the whole sum (which is just the determinant), must be no bigger than 120 in absolute value.

2. Problem 4.4.18. Find A^{-1} from the cofactor formula $C^T / \det A$. Use symmetry in part (b):

   $$\text{(a) } A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}. \qquad \text{(b) } A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

   (a) We need to determine the various cofactors of A to find the cofactor matrix C:

   $$C_{11} = (-1)^{1+1} \det A_{11} = \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3$$

   $$C_{12} = (-1)^{1+2} \det A_{12} = -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

   $$C_{13} = (-1)^{1+3} \det A_{13} = \begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0$$

   $$C_{21} = (-1)^{2+1} \det A_{21} = -\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = -2$$

$$C_{22} = (-1)^{2+2} \det A_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{23} = (-1)^{2+3} \det A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = -4$$

$$C_{31} = (-1)^{3+1} \det A_{31} = \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^{3+2} \det A_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^{3+3} \det A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

Therefore,

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -4 \\ 0 & 0 & 3 \end{bmatrix}.$$

Also, since we've already computed the relevant cofactors, it's easy to find $\det A$:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 1 \cdot 3 + 2 \cdot 0 + 0 \cdot 0 = 3.$$

Therefore,

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{3} \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -4/3 & 1 \end{bmatrix}$$

(b) Again, we need to determine the various cofactors of A to find the matrix C, but now we can use the fact that A is symmetric, which implies that $A_{ij} = A_{ji}^T$ and so

$$C_{ij} = (-1)^{i+j} \det A_{ij} = (-1)^{i+j} \det A_{ji}^T = (-1)^{j+i} \det A_{ji} = C_{ji}.$$

Thus, we only need to compute the C_{ij} for which $i \leq j$:

$$C_{11} = (-1)^{1+1} \det A_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$C_{12} = (-1)^{1+2} \det A_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2$$

$$C_{13} = (-1)^{1+3} \det A_{13} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$C_{22} = (-1)^{2+2} \det A_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$C_{23} = (-1)^{2+3} \det A_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2$$

$$C_{33} = (-1)^{3+3} \det A_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

Therefore,

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Also,

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2 \cdot 3 + (-1) \cdot 2 + 0 \cdot 1 = 4,$$

so we have that

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}.$$

3. Problem 4.4.24. If all entries of A are integers, and $\det A = 1$ or -1, prove that all entries of A^{-1} are integers. Give a 2 by 2 example.

   Proof. If all entries of A are integers, then every cofactor of A is an integers (since each cofactor is, ultimately, a sum of products of integers). Therefore, the entries in the cofactor matrix C are all integers, which means that the entries in C^T are also integers. Moreover, since $\det A = \pm 1$, we see that

   $$A^{-1} = \frac{1}{\det A} C^T = \pm C^T,$$

   so all of the entries of A^{-1} are integers. $\square$

   Example: Consider the 2 by 2 matrix

   $$A = \begin{bmatrix} 25 & 36 \\ 9 & 13 \end{bmatrix}$$

   Then $\det A = 25 \cdot 13 - 36 \cdot 9 = 325 - 324 = 1$, and so

   $$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 13 & -36 \\ -9 & 25 \end{bmatrix} = \begin{bmatrix} 13 & -36 \\ -9 & 25 \end{bmatrix},$$

   which is certainly a matrix with integer entries.

4. Problem 4.4.28. A box has edges from $(0,0,0)$ to $(3,1,1)$, $(1,3,1)$, and $(1,1,3)$. Find its volume and also find the area of each parallelogram face.

   Answer: If we form the matrix

   $$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix},$$

   then the columns of A form the edges of the box, and so the determinant of A will give the volume of the box:

   $$\begin{aligned} \det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 3 \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\ &= 3 \cdot 8 - 1 \cdot 2 + 1 \cdot (-2) \\ &= 20. \end{aligned}$$

Therefore, the volume of the box is 20.

Notice that, if we can find a vector $\vec{v}$ of length 1 which is perpendicular to the side spanned by the first two columns of A (call them $\vec{a}_1$ and $\vec{a}_2$), then the volume of the box spanned by $\vec{a}_1, \vec{a}_2$, and $\vec{v}$ will be the same as the area of the parallelogram spanned by $\vec{a}_1$ and $\vec{a}_2$. To find such a $\vec{v}$, note that the nullspace of the matrix

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

is orthogonal to its row space, which is just the subspace spanned by $\vec{a}_1$ and $\vec{a}_2$. Doing elimination on the above matrix yields

$$\begin{bmatrix} 3 & 0 & 3/4 \\ 0 & 8/3 & 2/3 \end{bmatrix},$$

so the homogeneous solutions (i.e. vectors in the nullspace) are of the form

$$x_3 \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \end{bmatrix}$$

for some real number x_3. Such vectors have length

$$\sqrt{x_3^2\left[\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + 1^2\right]} = |x_3|\sqrt{\frac{1}{16} + \frac{1}{16} + 1} = |x_3|\sqrt{\frac{9}{8}} = \frac{3|x_3|}{2\sqrt{2}}.$$

Thus, letting $x_3 = \frac{2\sqrt{2}}{3}$, we get

$$\begin{bmatrix} -\frac{\sqrt{2}}{6} \\ -\frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix}$$

which has length 1 and is perpendicular to $\vec{a}_1$ and $\vec{a}_2$. Therefore, the area of the parallelogram spanned by $\vec{a}_1$ and $\vec{a}_2$ is equal to

$$\begin{aligned} \det \begin{bmatrix} 3 & 1 & -\frac{\sqrt{2}}{6} \\ 1 & 3 & -\frac{\sqrt{2}}{6} \\ 1 & 1 & \frac{2\sqrt{2}}{3} \end{bmatrix} &= 3 \cdot \begin{vmatrix} 3 & -\frac{\sqrt{2}}{6} \\ 1 & \frac{2\sqrt{2}}{3} \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & -\frac{\sqrt{2}}{6} \\ 1 & \frac{2\sqrt{2}}{3} \end{vmatrix} + \frac{-\sqrt{2}}{6} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\ &= \frac{39\sqrt{2}}{6} - \frac{5\sqrt{2}}{6} + \frac{2\sqrt{2}}{6} \\ &= 6\sqrt{2} \\ &\approx 8.48. \end{aligned}$$

To find the area of the parallelograms spanned by $\vec{a}_1$ and $\vec{a}_3$ and by $\vec{a}_2$ and $\vec{a}_3$ we can use the exact same reasoning with a rearrangement of the components (since $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ are the same up to rearrangement of their components), so these other parallelograms also have area $6\sqrt{2}$.

5. Problem 4.4.32. If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4, what is the largest possible value for the determinant (based on volume)? If all entries are 1 or -1, what are those lengths and the maximum determinant?

   Answer: If the four columns have lengths L_1, L_2, L_3, L_4, then the four-dimensional box spanned by the columns of the matrix has edges of length L_1, L_2, L_3, L_4. Thus, the volume of this box is no larger than the product $L_1L_2L_3L_4$. Since the (absolute value of the) determinant of a matrix is equal to the volume of the box spanned by the columns, this implies that

$$|\det A| \leq L_1L_2L_3L_4.$$

   If all entries of the matrix are 1 or -1, then each L_i is equal to

$$\sqrt{(\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2} = \sqrt{4} = 2,$$

   and so

$$|\det A| \leq 2^4 = 16.$$

   In fact, the following is such a matrix with determinant exactly equal to 16:

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

6. Problem 5.1.8. Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda),$$

   and making a clever choice of λ.

   Proof. Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix A. Then the λ_i are the roots of the polynomial

$$\det(A - \lambda I),$$

   meaning that this polynomial factors as

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

   Letting λ (which we're thinking of purely as an abstract variable) equal zero and simplifying both sides of the above equation, we see that

$$\det A = \lambda_1\lambda_2\cdots\lambda_n,$$

   so the determinant of A is equal to the product of the eigenvalues of A. □

7. Problem 5.1.14. Find the rank and all four eigenvalues for both the matrix of ones and the checkerboard matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Which eigenvectors correspond to nonzero eigenvalues?

Answer: Since all columns of A are equal, the rank of A is equal to 1; this implies that 0 is an eigenvalue of A, with eigenvectors given by the elements of the nullspace of A. Since elements of the nullspace of A are of the form

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

we see that $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent eigenvectors corresponding to the eigenvalue 0.

To find the other eigenvalue of A (since A has rank 1, there can be at most one), we solve

$$\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\
&\qquad + 1 \cdot \begin{vmatrix} 1 & 1-\lambda & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \\ 1 & 1 & 1 \end{vmatrix} \\
&= (1-\lambda) \left((1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} \right) \\
&\quad - \left(1 \cdot \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} \right) \\
&\quad + \left(1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} - (1-\lambda) \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right) \\
&\quad - \left(1 \cdot \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} - (1-\lambda) \cdot \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right)
\end{aligned}$$

Then, after simplifying, we see that

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (1-\lambda)^4 - 6(1-\lambda)^2 + 8(1-\lambda) - 3 \\ &= \lambda^4 - 4\lambda^3 \\ &= \lambda^3(\lambda - 4). \end{aligned}$$

Therefore, the nonzero eigenvalue of A is 4; the corresponding eigenvector is in the nullspace of

$$A - 4I = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}.$$

This matrix reduces to

$$\begin{bmatrix} -3 & 0 & 0 & 3 \\ 0 & -8/3 & 0 & 8/3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the nullspace is the line containing the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, meaning that this is the eigenvector associated to the eigenvalue 4.

Turning to the checkerboard matrix C, notice that the first two columns are linearly independent, but that the third and fourth columns are repeats of the first two columns, so the matrix must have rank 2. Therefore, one eigenvalue is 0; the corresponding eigenvectors will be the elements of the nullspace of C. Since C reduces to

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ forms a basis for the nullspace of C, meaning that these two vectors are eigenvectors corresponding to the eigenvalue 0.

The other two eigenvalues will come from solving

$$\begin{aligned}
0 &= \det(C - \lambda I) \\
&= \begin{vmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{vmatrix} \\
&= -\lambda \cdot \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} + 0 - 1 \cdot \begin{vmatrix} 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ 1 & 0 & 1 \end{vmatrix} \\
&= -\lambda\left(-\lambda \cdot \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -\lambda \end{vmatrix} + 0\right) - \left(1 \cdot \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & -\lambda \end{vmatrix} + 0\right) \\
&\qquad + 0 - \left(1 \cdot \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} + \lambda \cdot \begin{vmatrix} 0 & -\lambda \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\right) \\
&= \lambda^4 - 4\lambda^2 \\
&= \lambda^2(\lambda^2 - 4) \\
&= \lambda^2(\lambda - 2)(\lambda + 2),
\end{aligned}$$

so the nonzero eigenvalues of C are 2 and -2. For the eigenvalue 2, the corresponding eigenvector will be any vector in the nullspace for

$$C - 2I = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}.$$

You can easily check that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is such an eigenvector.

Likewise, the eigenvector corresponding to the eigenvalue -2 will be any vector in the nullspace for

$$C - (-2)I = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

You can check that $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ is such an eigenvector.

8. Problem 5.1.24. What do you do to $A\vec{x} = \lambda\vec{x}$ in order to prove the following?

   (a) λ^2 is an eigenvalue of A^2.

Proof. Since we know that $A\vec{x} = \lambda\vec{x}$, we can just multiply both sides of this equation by A:

$$A^2\vec{x} = A(\lambda\vec{x}) = \lambda A\vec{x} = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$$

where in the second-to-last equality we used the fact that $A\vec{x} = \lambda\vec{x}$. Thus, we see that $\vec{x}$ is an eigenvector for the matrix A^2 with corresponding eigenvalue λ^2, so indeed λ^2 is an eigenvalue for A^2. $\square$

(b) λ^{-1} is an eigenvalue of A^{-1}.

Proof. Since we know $A\vec{x} = \lambda\vec{x}$ we can (assuming A is invertible) multiply both sides of this equation by A^{-1}, yielding

$$\vec{x} = A^{-1}(\lambda\vec{x}) = \lambda A^{-1}\vec{x}.$$

Since A is invertible, $\lambda \neq 0$, so we can multiply both sides by $\lambda^{-1} = \frac{1}{\lambda}$ and see that

$$\lambda^{-1}\vec{x} = A^{-1}\vec{x}.$$

But this just means that λ^{-1} is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$. $\square$

(c) $\lambda + 1$ is an eigenvalue of $A + I$.

Proof. We know that $A\vec{x} = \lambda\vec{x}$ and of course $I\vec{x} = \vec{x}$, so we have that

$$(A + I)\vec{x} = A\vec{x} + I\vec{x} = \lambda\vec{x} + \vec{x} = (\lambda + 1)\vec{x}.$$

Therefore, $\lambda + 1$ is an eigenvalue for $A + I$ with corresponding eigenvector $\vec{x}$. $\square$

9. Problem 5.1.26. Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i\sin\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotates the } xy\text{-plane by the angle } \theta.$$

Find the eigenvectors of Q by solving $(Q - \lambda I)\vec{x} = \vec{0}$. Use $i^2 = -1$.

Answer: We have that

$$\begin{aligned}
\det(Q - \lambda I) &= \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} \\
&= (\cos\theta - \lambda)(\cos\theta - \lambda) - (\sin\theta)(-\sin\theta) \\
&= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta \\
&= \lambda^2 - 2\lambda\cos\theta + \sin^2\theta + \cos^2\theta \\
&= \lambda^2 - 2\lambda\cos\theta + 1.
\end{aligned}$$

Therefore,

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\sin\theta,$$

so indeed the two complex numbers $\cos\theta \pm i\sin\theta$ are the two eigenvalues of Q. Now, the eigenvector corresponding to the eigenvalue $\cos\theta + i\sin\theta$ will be in the nullspace of the matrix

$$Q - (\cos\theta + i\sin\theta)I = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}.$$

To get this matrix into reduced echelon form, subtract i times row 1 from row 2 to get

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}.$$

Thus, the nullspace is the line spanned by the vector

$$\begin{bmatrix} i \\ 1 \end{bmatrix},$$

which is thus the eigenvector corresponding to the eigenvalue $\cos\theta + i\sin\theta$.

Similarly, the eigenvector corresponding to the eigenvalue $\cos\theta - i\sin\theta$ will be in the nullspace of the matrix

$$Q - (\cos\theta - i\sin\theta)I = \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}.$$

Adding i times row 1 to row 2 yields

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}.$$

Thus, the nullspace is the line spanned by the vector

$$\begin{bmatrix} -i \\ 1 \end{bmatrix},$$

which is thus the eigenvector corresponding to the eigenvalue $\cos\theta - i\sin\theta$.

10. Problem 5.1.30. Choose the second row of $A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}$ so that A has eigenvalues 4 and 7.

Answer: If A has eigenvalues 4 and 7, then it must be the case that

$$\det(A - \lambda I) = (\lambda - 4)(\lambda - 7) = \lambda^2 - 11\lambda + 28.$$

On the other hand, we know that, if $A = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ a_{21} & a_{22} - \lambda \end{vmatrix} = -\lambda(a_{22} - \lambda) - a_{21} = \lambda^2 - a_{22}\lambda - a_{21}.$$

Therefore, we should pick $a_{22} = 11$ and $a_{21} = -28$, meaning that

$$A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}.$$