Math 215 HW #6 Solutions

1. Problem 3.1.14. Show that $\mathbf{x} - \mathbf{y}$ is orthogonal to $\mathbf{x} + \mathbf{y}$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.

   Proof. First, suppose $\mathbf{x} - \mathbf{y}$ is orthogonal to $\mathbf{x} + \mathbf{y}$. Then

$$\begin{aligned} 0 &= \langle \mathbf{x} - \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= (\mathbf{x} - \mathbf{y})^T(\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x}^T\mathbf{x} + \mathbf{x}^T\mathbf{y} - \mathbf{y}^T\mathbf{x} - \mathbf{y}^T\mathbf{x} \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle - \rangle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle \end{aligned}$$

   since $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$. In other words,

$$0 = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2,$$

   so $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2$. Since the norm of a vector can never be negative, this implies that

$$\|\mathbf{x}\| = \|\mathbf{y}\|.$$

   Thus, we see that if $\mathbf{x} - \mathbf{y}$ is orthogonal to $\mathbf{x} + \mathbf{y}$, then $\|\mathbf{x}\| = \|\mathbf{y}\|$.

   On the other hand, suppose $\|\mathbf{x}\| = \|\mathbf{y}\|$. Then

$$\begin{aligned} \langle \mathbf{x} - \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle &= (\mathbf{x} - \mathbf{y})^T(\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x}^T\mathbf{x} + \mathbf{x}^T\mathbf{y} - \mathbf{y}^T\mathbf{x} - \mathbf{y}^T\mathbf{x} \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle - \rangle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 \\ &= 0, \end{aligned}$$

   so we see that $\mathbf{x} - \mathbf{y}$ is orthogonal to $\mathbf{x} + \mathbf{y}$.

   We've seen that the implication goes both ways, so we conclude that $\mathbf{x} - \mathbf{y}$ is orthogonal to $\mathbf{x} + \mathbf{y}$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$, as desired. □

2. Problem 3.1.20. Let $\mathbf{S}$ be a subspace of $\mathbb{R}^n$. Explain what $(\mathbf{S}^\perp)^\perp = \mathbf{S}$ means and why it is true.

   Answer: First, $(\mathbf{S}^\perp)^\perp$ is the orthogonal complement of $\mathbf{S}^\perp$, which is itself the orthogonal complement of $\mathbf{S}$, so $(\mathbf{S}^\perp)^\perp = \mathbf{S}$ means that $\mathbf{S}$ is the orthogonal complement of its orthogonal complement.

   To show that it is true, we want to show that $\mathbf{S}$ is contained in $(\mathbf{S}^\perp)^\perp$ and, conversely, that $(\mathbf{S}^\perp)^\perp$ is contained in $\mathbf{S}$; if we can show both containments, then the only possible conclusion is that $(\mathbf{S}^\perp)^\perp = \mathbf{S}$.

   To show the first containment, suppose $\mathbf{v} \in \mathbf{S}$ and $\mathbf{w} \in \mathbf{S}^\perp$. Then

$$\langle \mathbf{v}, \mathbf{w} \rangle = 0$$

by the definition of $\mathbf{S}^\perp$. Thus, $\mathbf{S}$ is certainly contained in $(\mathbf{S}^\perp)^\perp$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $\mathbf{S}^\perp$).

To show the other containment, suppose $\mathbf{v} \in (\mathbf{S}^\perp)^\perp$ (meaning that $\mathbf{v}$ is orthogonal to all vectors in $\mathbf{S}^\perp$); then we want to show that $\mathbf{v} \in \mathbf{S}$. I'm sure there must be a better way to see this, but here's one that works. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a basis for $\mathbf{S}$ and let $\{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$ be a basis for $\mathbf{S}^\perp$. If $\mathbf{v} \notin \mathbf{S}$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}\}$ is a linearly independent set. Since each vector in that set is orthogonal to all of $\mathbf{S}^\perp$, the set

$$\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$$

is linearly independent. Since there are $p+q+1$ vectors in this set, this means that $p+q+1 \leq n$ or, equivalently, $p+q \leq n-1$. On the other hand, if A is the matrix whose ith row is u_i^T, then the row space of A is $\mathbf{S}$ and the nullspace of A is $\mathbf{S}^\perp$. Since $\mathbf{S}$ is p-dimensional, the rank of A is p, meaning that the dimension of $\text{nul}(A) = \mathbf{S}^\perp$ is $q = n-p$. Therefore,

$$p + q = p + (n-p) = n,$$

contradicting the fact that $p + q \leq n - 1$. From this contradiction, then, we see that, if $\mathbf{v} \in (\mathbf{S}^\perp)^\perp$, it must be the case that $\mathbf{v} \in \mathbf{S}$.

3. Problem 3.1.28. This is a system of equations $A\mathbf{x} = \mathbf{b}$s with *no solution*:

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9. \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector $\mathbf{y}$ in which subspace? The inner product is $\mathbf{y}^T\mathbf{b}$ is 1.

Answer: Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix}.$$

Then the given system is equivalent to the matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Then multiplying the ith row in the system by y_i and adding is equivalent to multiplying both sides of the matrix equation by $\mathbf{y}^T$. We see that

$$\begin{aligned}\mathbf{y}^T A\mathbf{x} &= [1\ 1\ -1]\begin{bmatrix}1 & 2 & 2\\ 2 & 2 & 3\\ 3 & 4 & 5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}\\ &= [1\ 1\ -1]\begin{bmatrix}x+2y+2z\\ 2x+2y+3z\\ 3x+4y+5z\end{bmatrix}\\ &= (x+2y+2z)+(2x+2y+3z)-(3x+4y+5z)\\ &= 0.\end{aligned}$$

On the other hand,

$$\mathbf{y}^T\mathbf{b} = [1\ 1\ -1]\begin{bmatrix}5\\ 5\\ 9\end{bmatrix} = 5+5-9 = 1.$$

Therefore, $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T\mathbf{b}$ reduces to

$$0 = 1,$$

so we see that the system has no solution. Since

$$0 = \mathbf{y}^T A\mathbf{x} = \langle \mathbf{y}, A\mathbf{x}\rangle,$$

we see that $\mathbf{y}$ is perpendicular to $A\mathbf{x}$ no matter what $\mathbf{x}$ is. Therefore, the vector $\mathbf{y}$ is perpendicular to the column space of A. Since the orthogonal complement of $\text{col}(A)$ is the left nullspace of A, we see that $\mathbf{y}$ must be an element of the left nullspace of A.

4. Problem 3.1.36. Extend Problem 35 to a p-dimensional subspace $\mathbf{V}$ and a q-dimensional subspace $\mathbf{W}$ of $\mathbb{R}^n$. What inequality on $p+q$ guarantees that $\mathbf{V}$ intersects $\mathbf{W}$ in a nonzero vector? These subspaces cannot be orthogonal.

   (Restatement: Suppose $\mathbf{V}$ is a p-dimensional subspace of $\mathbb{R}^n$ and that $\mathbf{W}$ is a q-dimensional subspace of $\mathbb{R}^n$. Moreover, suppose that $\mathbf{V}$ and $\mathbf{W}$ are orthogonal as subspaces of $\mathbb{R}^n$. What must be true of the quantity $p+q$?)

   Answer: I claim that if $p+q>n$, then $\mathbf{V}$ and $\mathbf{W}$ intersect in some nonzero vector. Equivalently, if $\mathbf{V}$ and $\mathbf{W}$ are orthogonal subspaces, then $p+q\leq n$.

   To see this, suppose $p+q>n$. Let $\{\mathbf{v}_1,\dots,\mathbf{v}_p\}$ be a basis for $\mathbf{V}$ and $\{\mathbf{w}_1,\dots,\mathbf{w}_q\}$ be a basis for $\mathbf{W}$. Then, since $\mathbb{R}^n$ is n-dimensional and since $p+q>n$, it must be the case that

$$\{\mathbf{v}_1,\dots,\mathbf{v}_p,\mathbf{w}_1,\dots,\mathbf{w}_q\}$$

   is a linearly dependent set. Thus, there exist real numbers $c_1,\dots,c_{p+q}$ not all zero such that

$$c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p+c_{p+1}\mathbf{w}_1+\ldots+c_{p+q}\mathbf{w}_q = 0.$$

   Stated another way,

$$c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p = -c_{p+1}\mathbf{w}_1-\ldots-c_{p+q}\mathbf{w}_q.$$

   The vector on the left-hand side is clearly an element of $\mathbf{V}$, whereas the vector on the right-hand side is just as clearly an element of $\mathbf{W}$. Therefore, the above vector (which is necessarily nonzero) is an element of the intersection $\mathbf{V}\cap\mathbf{W}$.

5. Problem 3.1.42. Suppose $\mathbf{S}$ is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $\mathbf{S}^\perp$. This is the same as solving $A\mathbf{x}0\mathbf{o}$ for which A?

   Answer: Since the given vectors live in $\mathbb{R}^4$ and are linearly independent, we see that $\mathbf{S}$ is 2-dimensional. Thus, it makes sense that $\mathbf{S}^\perp$ should also be 2-dimensional.

   Since we know that the orthogonal complement of the row space of any matrix is the nullspace of the matrix, we can determine $\mathbf{S}^\perp$ simply by constructing a matrix A whose row space is $\mathbf{S}$ and then determining the nullspace of A. Clearly, if

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$$

   then the row space of A is equal to $\mathbf{S}$. To find the nullspace of A, we want to row reduce the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 1 & 3 & 3 & 2 & 0 \end{array}\right].$$

   First, subtract row 1 from row 2:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right].$$

   Next, subtract twice row 2 from row 1:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right].$$

   The corresponding system of equations is

$$\begin{aligned} x_1 + 5x_4 &= 0 \\ x_2 + x_3 - x_4 &= 0 \end{aligned}$$

   or, equivalently,

$$\begin{aligned} x_1 &= -5x_4 \\ x_2 &= -x_3 + x_4. \end{aligned}$$

   Hence, the solutions to the equation $A\mathbf{x} = \mathbf{0}$ are of the form

$$x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Thus,

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   is a basis for the nullspace of A, i.e. a basis for $\mathbf{S}^\perp$.

6. Problem 3.1.46. Find $A^T A$ if the columns of A are unit vectors, all mutually perpendicular.

   Answer: Suppose A is $m \times n$, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then

$$A^T A = \begin{bmatrix} — \mathbf{a}_1^T — \\ — \mathbf{a}_2^T — \\ \vdots \\ — \mathbf{a}_n^T — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \ldots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \ldots & \mathbf{a}_2^T\mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \ldots & \mathbf{a}_n^T\mathbf{a}_n \end{bmatrix}.$$

   In other words, the (i, j) entry of the matrix $A^T A$ is equal to

$$\mathbf{a}_i^T \mathbf{a}_j = \langle \mathbf{a}_i, \mathbf{a}_j \rangle.$$

   Since the columns are all mutually perpendicular, the above inner product will be zero unless $i = j$, meaning that the only nonzero entries in $A^T A$ will be on the diagonal.

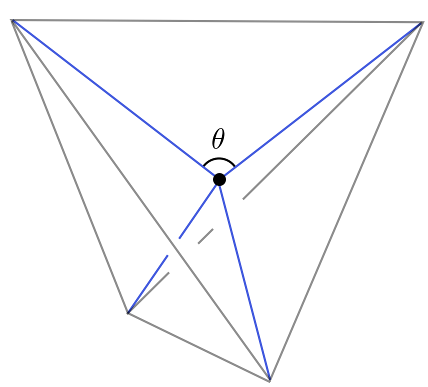
   Moreover, $\langle \mathbf{a}_i, \mathbf{a}_i \rangle = \|\mathbf{a}_i\|^2 = 1$ since each column is a unit vector. Therefore,

$$A^T A = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix},$$

   the identity matrix.

7. Problem 3.2.8. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$—note that all six edges have length $\sqrt{2}$, so the tetrahedron is regular—what is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices? (The bond angle itself is about 109.5°, an old friend of chemists).

   Answer:



   In the above figure, the black dot is the carbon atom and the blue segments are the bonds between carbon and hydrogen atoms (the gray segments are the edges of the tetrahedron and don't mean anything chemically). The goal is to determine the cosine of the angle θ between two of the blue segments.

The top left blue segment corresponds to the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix},$$

whereas the top right blue segment corresponds to the vector

$$\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

We know that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta,$$

so

$$\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}.$$

The numerator is

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \mathbf{w} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = -\frac{1}{4} - \frac{1}{4} + \frac{1}{4} = -\frac{1}{4}.$$

To find the first term in the denominator, note that

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \mathbf{v}^T \mathbf{v} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4},$$

so

$$\|\mathbf{v}\| = \sqrt{\|\mathbf{v}\|^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

A similar calculation shows that

$$\|\mathbf{w}\| = \frac{\sqrt{3}}{2}$$

as well, so

$$\|\mathbf{v}\|\|\mathbf{w}\| = \frac{3}{4}.$$

Therefore, putting it all together,

$$\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-1/4}{3/4} = -\frac{1}{3}.$$

8. Problem 3.2.14. What matrix P projects every point of $\mathbb{R}^3$ onto the line of intersection of the planes $x+y+t=0$ and $x-t=0$?

Answer: First, we need to find a vector on the line of intersection. Note that any such vector is necessarily a solution of the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To solve this equation, we'll do Gaussian elimination on the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

Subtract row 1 from row 2 to get

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix},$$

Then add row 2 to row 1 and multiply row 2 by -1:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

This corresponds to the system of equations

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3. \end{aligned}$$

Therefore, solutions to the matrix equation are of the form

$$x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

meaning that the vector $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in the desired line of intersection.

Now, just recall that the projection matrix P is given by

$$P = \frac{\mathbf{a}\,\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}.$$

The denominator is simply

$$\mathbf{a}^T\mathbf{a} = [1 \ \ -2 \ 1] \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 + 4 + 1 = 6.$$

The numerator is

$$\mathbf{a}\,\mathbf{a}^T = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} [1 \ \ -2 \ 1] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

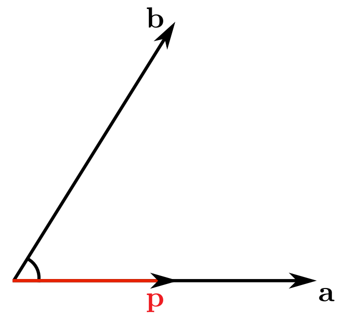
Therefore,

$$P = \frac{\mathbf{a}\,\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

9. Problem 3.2.18. *Draw* the projection of $\mathbf{b}$ onto $\mathbf{a}$ and also compute it from $\mathbf{p} = \widehat{\mathbf{x}}\mathbf{a}$:

$$\text{(a) } \mathbf{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \qquad \text{(b) } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Answer: (a)



Since $\widehat{\mathbf{x}} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}$, we just compute

$$\mathbf{a}^T\mathbf{b} = [1\ 0] \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos\theta$$

and

$$\mathbf{a}^T\mathbf{a} = [1\ 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1,$$

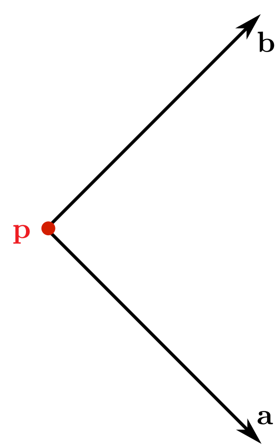
so

$$\widehat{\mathbf{x}} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}} = \frac{\cos\theta}{1} = \cos\theta.$$

Therefore,

$$\mathbf{p} = \widehat{\mathbf{x}}\mathbf{a} = \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}.$$

(b)



Since $\widehat{\mathbf{x}} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}$, we just compute

$$\mathbf{a}^T\mathbf{b} = [1 \ \ -1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 - 1 = 0$$

and

$$\mathbf{a}^T\mathbf{a} = [1 \ \ -1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 + 1 = 2,$$

so

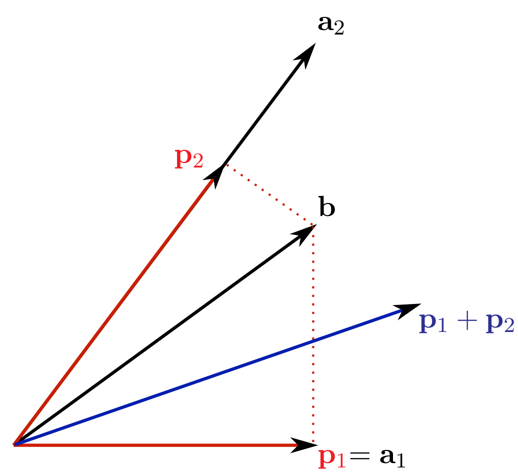
$$\widehat{\mathbf{x}} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}} = \frac{0}{2} = 0.$$

Therefore,

$$\mathbf{p} = \widehat{\mathbf{x}}\mathbf{a} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

10. Problem 3.2.24. Project the vector $\mathbf{b} = (1, 1)$ onto the lines through $\mathbf{a}_1 = (1, 0)$ and $\mathbf{a}_2 = (1, 2)$. Draw the projections $\mathbf{p}_1$ and $\mathbf{p}_2$ and add $\mathbf{p}_1 + \mathbf{p}_2$. The projections do not add to $\mathbf{b}$ because the $\mathbf{a}$'s are not orthogonal.

Answer:



First, notice that the projection of $\mathbf{b}$ onto the line through $\mathbf{a}_1$ is

$$\mathbf{p}_1 = \frac{\mathbf{a}_1^T\mathbf{b}}{\mathbf{a}_1^T\mathbf{a}_1}\mathbf{a}.$$

Since

$$\mathbf{a}_1^T\mathbf{b} = [1 \ 0]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 + 0 = 1$$

and

$$\mathbf{a}_1^T\mathbf{a}_1 = [1 \ 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + 0 = 1,$$

we have that

$$\mathbf{p}_1 = \frac{\mathbf{a}_1^T\mathbf{b}}{\mathbf{a}_1^T\mathbf{a}_1}\mathbf{a}_1 = \frac{1}{1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Next, the projection of $\mathbf{b}$ onto the line through $\mathbf{a}_2$ is

$$\mathbf{p}_2 = \frac{\mathbf{a}_2^T \mathbf{b}}{\mathbf{a}_2^T \mathbf{a}_2} \mathbf{a}_2.$$

Since

$$\mathbf{a}_2^T \mathbf{b} = [1\ 2] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 + 2 = 3$$

and

$$\mathbf{a}_2^T \mathbf{a}_2 = [1\ 2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 + 4 = 5,$$

we have that

$$\mathbf{p}_2 = \frac{\mathbf{a}_2^T \mathbf{b}}{\mathbf{a}_2^T \mathbf{a}_2} \mathbf{a}_2 = \frac{3}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}.$$

Therefore,

$$\mathbf{p}_1 + \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 6/5 \end{bmatrix}$$