

Math 215 HW #4 Solutions

1. Problem 2.3.6. Choose three independent columns of U . Then make two other choice. Do the same for A . You have found bases for which spaces?

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

Solution: The most obvious way to choose three independent columns of U is to pick the three pivot columns, namely

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix}.$$

Another choice of three independent columns would be the first, third and fourth columns, which are independent because, if

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 7 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_1 + 4c_2 + c_3 \\ 7c_2 \\ 9c_3 \\ 0 \end{bmatrix},$$

then it must be the case that $c_2 = c_3 = 0$ (from the second and third components), so the first component reduces to $0 = 2c_1$, so $c_1 = 0$ as well.

We could also have chosen the second, third, and fourth columns. To see that these are independent, suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 7 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 3c_1 + 4c_2 + c_3 \\ 6c_1 + 7c_2 \\ 9c_3 \\ 0 \end{bmatrix}.$$

Then the third component implies that $c_3 = 0$ and the first two component give the system of equations

$$\begin{aligned} 3c_1 + 4c_2 &= 0 \\ 6c_1 + 7c_2 &= 0. \end{aligned}$$

Interpreting via the augmented matrix $\begin{bmatrix} 3 & 4 & 0 \\ 6 & 7 & 0 \end{bmatrix}$ and reducing by subtracting twice row 1 from row 2 yields

$$\begin{bmatrix} 3 & 4 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Then $c_2 = 0$ and, since $3c_1 + 4c_2 = 0$, this means that $c_1 = 0$ as well. So, as indicated, the second, third, and fourth columns of U are also independent.

Since U is just the echelon form of A (in particular, you can get from A to U by subtracting twice row 1 from row 4), the relationship between the columns of A is the same as the relationship between the columns of U . Hence, we can translate the three choices made above into three different choices of independent columns of A . Namely, the first, second, and fourth columns of A are independent, as are the first, third, and fourth columns of A , and the second, third, and fourth columns of A are also independent.

2. Problem 2.3.10. Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in \mathbb{R}^4 . Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Solution: Notice that

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

are vectors in the plane $x + 2y - 3z - t = 0$. They're independent since

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_1 + 3c_2 \\ -c_1 \\ c_2 \\ 0 \end{bmatrix}$$

implies that $c_1 = 0 = c_2$. The vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is also in this plane and we can see that this

collection of three vectors is linearly independent as follows: suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 + 3c_2 + c_3 \\ -c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Then it must be the case that $c_1 = c_2 = c_3 = 0$, so we see that the vectors

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent vectors in the plane $x + 2y - 3z - t = 0$ in \mathbb{R}^4 .

There cannot be four linearly independent vectors in this plane because any collection of four linearly independent vectors in \mathbb{R}^4 must span all of \mathbb{R}^4 . Since there are clearly vectors in \mathbb{R}^4

that are not in the plane (for example, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$), this cannot be the case.

Any vector in the plane $x + 2y - 3z - t = 0$ is a solution to the matrix equation

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and, *vice versa*, any solution to this matrix equation definitely lies in the plane. Therefore, the plane $x + 2y - 3z - t = 0$ is the nullspace of the matrix

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \end{bmatrix}.$$

(Incidentally, this provides another check that there can't be four linearly independent vectors in the plane. Since this matrix clearly has rank 1, we know that the dimension of the nullspace is $4 - 1 = 3$, so the plane $x + 2y - 3z - t = 0$, which is the same as the nullspace, is also three-dimensional and so cannot contain four linearly independent vectors)

3. Problem 2.3.26. Suppose \mathbf{S} is a five-dimensional subspace of \mathbb{R}^6 . True or false?

(a) Every basis for \mathbf{S} can be extended to a basis for \mathbb{R}^6 by adding one more vector.

Answer: True. Just pick any vector in \mathbb{R}^6 that is linearly independent from the given basis (there must be lots of them, since \mathbb{R}^6 is 6-dimensional and \mathbf{S} is 5-dimensional). Then the set consisting of the given basis plus this new vector is, by construction, linearly independent and spans a 6-dimensional space, so it must span all of \mathbb{R}^6 . Any set of linearly independent vectors that spans all of \mathbb{R}^6 is a basis for \mathbb{R}^6 , so this is indeed a basis for \mathbb{R}^6 .

(b) Every basis for \mathbb{R}^6 can be reduced to a basis for \mathbf{S} by removing one vector.

Answer: False. Suppose \mathbf{S} is the five-dimensional subspace described by the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0.$$

(One way to see this is both a subspace and five-dimensional: by the same reasoning as in Problem 2, it is the nullspace of the 6×6 matrix with all entries equal to 1.) Consider the standard basis for \mathbb{R}^6 :

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Then none of the elements of this basis lies in \mathbf{S} , so deleting one of them cannot possibly give a basis for \mathbf{S} .

4. Problem 2.3.38.

- (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

Solution: Integrating both sides of the equation, we see that

$$y(x) = 0 + C = C,$$

where C is some constant. Thus, the functions that satisfy $\frac{dy}{dx} = 0$ are exactly the constant functions.

- (b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

Solution: Notice that $\frac{d}{dx}(3x) = 3$, so $y_p(x) = 3x$ is a particular function that solves $\frac{dy}{dx} = 3$.

- (c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

Solution: I claim that all functions that satisfy $\frac{dy}{dx} = 3$ are of the form

$$y(x) = y_p(x) + C = 3x + C.$$

On the one hand, this is completely obvious from our knowledge of Calculus. On the other hand, we can prove this just using parts (a) and (b) above.

In particular, suppose $y(x)$ is some solution to $\frac{dy}{dx} = 3$. Then

$$\frac{d}{dx}(y(x) - y_p(x)) = \frac{dy}{dx} - \frac{dy_p}{dx} = 3 - 3 = 0,$$

so $y(x) - y_p(x)$ is a solution to the equation $\frac{dy}{dx} = 0$. Hence, by part (a), we know that

$$y(x) - y_p(x) = C$$

for some constant C , so we see that

$$y(x) = y_p(x) + C = 3x + C.$$

5. Problem 2.4.10. If $Ax = b$ always has at least one solution, show that the only solution to $A^T y = 0$ is $y = 0$. *Hint:* What is the rank?

Solution: If A is an $m \times n$ matrix and $Ax = b$ always has at least one solution for any choice of $b \in \mathbb{R}^m$, that means that any vector $b \in \mathbb{R}^m$ must lie in the column space of A . In other words,

$$\text{col}(A) = \mathbb{R}^m.$$

This, then, means that the dimensions of these two spaces must be the same:

$$\dim \text{col}(A) = \dim \mathbb{R}^m = m.$$

Since the rank of A is the same as the dimension of the column space of A , we see that A has rank m .

Now, the solutions to the equation $A^T y = 0$ are precisely those vectors y which lie in $\text{nul}(A^T)$, otherwise known as the left nullspace of A . However, the dimension of the left nullspace of A is given by

$$\dim \text{nul}(A^T) = m - \text{rank}(A) = m - m = 0.$$

Therefore, $\text{nul}(A^T)$ is zero-dimensional, so it only contains the zero vector, so the zero vector is the only solution to the equation $A^T y = 0$.

6. Problem 2.4.28. Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

Solution: Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

Notice, first of all, that A is symmetric, so a basis for its column space will also be a basis for its row space (and *vice versa*). Notice that the second column is twice the first column minus twice the third column, so we can ignore the second column for the purposes of finding a basis for the column space.

Also, the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are certainly linearly independent since they aren't multiples of each other. Therefore, these two columns do indeed give a basis for the column space of A (and, by symmetry, for the row space of A).

Now, I claim that this cannot be a basis for the row space and the nullspace of any 3×3 matrix A . To see why, remember that the dimension of the row space of A is equal to the rank of A , whereas the dimension of the nullspace is equal to the number of columns minus the rank, namely

$$3 - \text{rank}(A).$$

If these two vectors are a basis for both the row space and the nullspace, then both the row space and the nullspace must be two-dimensional, meaning that

$$\text{rank}(A) = 2 \quad \text{and} \quad 3 - \text{rank}(A) = 2,$$

meaning that the rank of A must be both 2 and 1 simultaneously. This is clearly impossible, so we see that the row space and the nullspace can't both be two-dimensional, so no two vectors can be a basis for both the row space and the nullspace of any 3×3 matrix.

7. Problem 2.4.32. Describe the four subspace of \mathbb{R}^4 associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: The column space of A is just the space spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

namely the xy -plane.

Notice that A is already in reduced echelon form, corresponding to the equations $y = 0$ and $z = 0$. This leaves x as a free variable, so the solutions to the homogeneous equation are those vectors of the form

$$\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix},$$

so the nullspace of A is the x -axis.

The row space of A (i.e. the column space of A^T) is spanned by the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which is just the yz -plane.

The left nullspace of A (i.e. $\text{nul}(A^T)$) consists of those vectors y such that $A^T y = 0$. Since

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is already in reduced echelon form, we can read off the equations $x = 0$ and $y = 0$. Hence, z is a free variable, so the nullspace of A^T consists of those vectors of the form

$$\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}.$$

Thus, the left nullspace of A is the z -axis.

(Notice that the column space and the left nullspace are perpendicular; similarly, the row space and the nullspace are perpendicular.)

Now, turn to $I + A$. Notice that we don't have to do any elimination to see that $I + A$ has three pivot columns (and three pivot rows), so the rank of $I + A$ is equal to three. Thus, both the column space and the row space are three-dimensional, so each is all of \mathbb{R}^3 . Moreover,

$$\dim \text{nul}(A) = 3 - \text{rank}(A) = 3 - 3 = 0,$$

so the nullspace is zero-dimensional, meaning it consists only of the zero vector. Likewise,

$$\dim \text{nul}(A^T) = 3 - \text{rank}(A) = 3 - 3 = 0,$$

so the left nullspace is also zero-dimensional and only contains the zero vector.

8. Problem 2.6.6. What 3 by 3 matrices represent the transformations that

(a) project every vector onto the xy -plane?

Solution: To find the matrix for this transformation, we just need to apply the transformation to the standard basis for \mathbb{R}^3 : the i th column of the matrix will be whatever vector results from applying the transformation to \mathbf{e}_i .

Projecting the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to the xy -plane has no effect on it, so the first column of

the matrix is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Likewise, projecting $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to the xy -plane has no effect, so the second column of the

matrix is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Finally, projecting $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to the xy -plane yields $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so the third column of the matrix

is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Therefore, the matrix of this transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) reflect every vector through the xy -plane?

Solution: Again, we just apply the transformation to the standard basis.

Reflecting $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ through the xy -plane has no effect, so the first column of the matrix

is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Likewise, reflecting $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ through the xy -plane has no effect, so the second column of

the matrix is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Finally, reflecting $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ through the xy -plane yields $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, so the third column of

the matrix is $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

Therefore, the matrix of this transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(c) rotate the xy -plane through 90° , leaving the z -axis alone?

Solution: Applying the transformation to the standard basis, we see that rotating $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ through 90° (counterclockwise) yields $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so the first column of the matrix is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Rotating $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ through 90° yields $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, so the second column of the matrix is $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$.

Finally, the transformation leaves the z -axis alone, so $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is unchanged, and so is the third column.

Therefore, the matrix of this transformation is

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) rotate the xy -plane, then xz , then yz , through 90° ?

Solution: First, apply this transformation to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: the rotation in the xy -plane yields

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, the rotation in the xz -plane leaves this vector unaffected, and then the rotation

in the yz -plane yields $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore, the first column of the matrix is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Now, apply the transformation to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The rotation in the xy -plane yields $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

Rotating this vector in the xz -plane yields $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. Finally, rotating this vector in the

yz -plane yields $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so we see that the second column of the matrix is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (notice that this means that the transformation actually has no effect on the y -axis).

Finally, apply the transformation to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The rotation in the xy -plane has no effect on

this vector. Rotating in the xz -plane takes this vector to $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, and then rotation in

the yz -plane has no further effect. Therefore, the third column of the matrix is $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$.

Putting all of this together, the matrix of the transformation is

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(You can easily check that this is the same matrix you would get if the transformation had just been rotating the xz -plane by 90° and leaving the y -axis alone, so the rotations in the xy -plane and the yz -plane were actually superfluous).

(e) carry out the same three rotations, but each one through 180° ?

Solution: Rotating $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ through 180° in the xy -plane yields $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$. Rotating this

through 180° in the xz -plane gets us back to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which is, in turn, unaffected by a

180° rotation in the yz -plane. Thus, the first column of the matrix is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Rotating $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in the xy -plane yields $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, which is then unaffected by the 180°

rotation in the xz -plane. Rotating by 180° in the yz -plane gets us back to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, which

is, therefore, the second column of the matrix.

Finally, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is unaffected by a 180° rotation in the xy -plane. Rotating by 180° in the

xz -plane yields $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$; rotating this by 180° in the yz -plane takes us back to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

which is the third column of the matrix.

Thus, the matrix of the transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Notice that this means we could have achieved the same effect by doing *nothing*, since this is also the matrix of the identity transformation).

9. Problem 2.6.14. Prove that T^2 is a linear transformation if T is linear (from \mathbb{R}^3 to \mathbb{R}^3).

Proof. Suppose $x, y \in \mathbb{R}^3$ and that $a, b \in \mathbb{R}$. Then, in order to show that T^2 is linear, we need to show that $T^2(ax + by) = aT^2(x) + bT^2(y)$. To that end, notice that

$$T^2(ax + by) = T(T(ax + by)) = T(aT(x) + bT(y)),$$

since T is a linear transformations. In turn, again using the fact that T is a linear transformation, the above is equal to

$$aT(T(x)) + bT(T(y)) = aT^2(x) + bT^2(y).$$

Stringing these equalities together, we see that

$$T^2(ax + by) = aT^2(x) + bT^2(y),$$

so T^2 is indeed linear. □

10. Problem 2.6.22. Which of these transformations is not linear? The input is $v = (v_1, v_2)$.

- (a) $T(v) = (v_2, v_1)$.
- (b) $T(v) = (v_1, v_1)$.
- (c) $T(v) = (0, v_1)$.
- (d) $T(v) = (0, 1)$.

Solution: I claim that (d) is the only one of these transformations that is not linear. To see this, let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be two vectors and let $a, b \in \mathbb{R}$ be scalars.

To see that the transformation in (a) is linear, notice that

$$\begin{aligned} T(av + bw) &= T\left(a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + b \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} av_1 + bw_1 \\ av_2 + bw_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} av_2 + bw_2 \\ av_1 + bw_1 \end{bmatrix} \\ &= a \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} + b \begin{bmatrix} w_2 \\ w_1 \end{bmatrix} \\ &= aT(v) + bT(w). \end{aligned}$$

Thus, the T from (a) is a linear transformation.

Next, to see that the transformation from (b) is linear, we check that

$$\begin{aligned} T(av + bw) &= T\left(a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + b \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} av_1 + bw_1 \\ av_2 + bw_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} av_1 + bw_1 \\ av_1 + bw_1 \end{bmatrix} \\ &= a \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} + b \begin{bmatrix} w_1 \\ w_1 \end{bmatrix} \\ &= aT(v) + bT(w). \end{aligned}$$

Hence, the T from (b) is a linear transformation.

Turning to (c), we see that

$$\begin{aligned} T(av + bw) &= T\left(a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + b \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} av_1 + bw_1 \\ av_2 + bw_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 \\ av_1 + bw_1 \end{bmatrix} \\ &= a \begin{bmatrix} 0 \\ v_1 \end{bmatrix} + b \begin{bmatrix} 0 \\ w_1 \end{bmatrix} \\ &= aT(v) + bT(w), \end{aligned}$$

so this is also a linear transformation.

Finally, the transformation in (d) is not linear. To see this, notice that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (*)$$

whereas

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \quad (\dagger)$$

Since the expressions in $(*)$ and (\dagger) are not equal, the transformation T cannot be linear.