Math 215 HW #4 Solutions

1. Problem 2.1.6. Let $\mathbf{P}$ be the plane in 3-space with equation $x + 2y + z = 6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbb{R}^3$?

   Answer: For any real number r, the plane $x + 2y + z = r$ is parallel to $\mathbf{P}$, since all such planes have a common normal vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. In particular, notice that the plane determined by the equation

   $$x + 2y + z = 0 \qquad (*)$$

   is parallel to $\mathbf{P}$ and passes through the origin (since $(x, y, z) = (0, 0, 0)$ is a solution of the above equation). Hence, this is the equation which determines the plane $\mathbf{P}_0$.

   Now, suppose $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbf{P}_0$; i.e. the triples (x_1, y_1, z_1) and (x_2, y_2, z_2) both satisfy the equation $(*)$. Then

   $$(x_1 + x_2) + 2(y_1 + y_2) + (z_1 + z_2) = (x_1 + 2y_1 + z_1) + (x_2 + 2y_2 + z_2) = 0 + 0 = 0,$$

   so we have that

   $$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \in \mathbf{P}_0.$$

   Also, if $c \in \mathbb{R}$, then

   $$cx_1 + 2(cy_1) + cz_1 = c(x_1 + 2y_1 + z_1) = c(0) = 0,$$

   so

   $$c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix} \in \mathbf{P}_0.$$

   Therefore, $\mathbf{P}_0$ is a subspace of $\mathbb{R}^3$.

   On the other hand, $\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ are both in $\mathbf{P}$, but

   $$\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

   is not in $\mathbf{P}$ since

   $$6 + 2(3) + 0 = 12 \neq 6.$$

   Therefore, we see that $\mathbf{P}$ is not a subspace of $\mathbb{R}^3$.

2. Problem 2.1.12. The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in the vector space $\mathbf{F}$ of all real functions. The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ________. Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$?

   Answer: The combination $3f(x) - 4g(x)$ is the function

   $$h(x) = 3x^2 - 20x.$$

   If we tried to define scalar multiplication as $cf(x) = f(cx)$ we would run into problems. Note that

   $$f(5x) = (5x)^2 = 25x^2,$$

   but

   $$f(2x) + f(3x) = (2x)^2 + (3x)^2 = 4x^2 + 9x^2 = 13x^2.$$

   Hence, this attempted definition of scalar multiplication would not satisfy rule 8 in the definition of a vector space.

3. Problem 2.1.18.

   (a) The intersection of two planes through $(0, 0, 0)$ is probably a ________ but it could be a ________. It can't be the zero vector $\mathbf{Z}$!

   Answer: The intersection of two planes through the origin in $\mathbb{R}^3$ is probably a line, but it could be a plane (if the two planes coincide).

   (b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a ________ but it could be a ________.

   Answer: The intersection of a plane through the origin with a line through the origin in $\mathbb{R}^3$ is probably just the single point $(0, 0, 0)$, but it could be a whole line (if the line lies in the plane).

   (c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbb{R}^5$, their intersection $\mathbf{S} \cap \mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbb{R}^5$. *Check the requirements on $x + y$ and cx.*

   Answer: To see that $\mathbf{S} \cap \mathbf{T}$ is a subspace, suppose $x, y \in \mathbf{S} \cap \mathbf{T}$ and that $c \in \mathbb{R}$. Then, since x and y are both in $\mathbf{S}$ and since $\mathbf{S}$ is a subspace (meaning that it is closed under addition), we have that

   $$x + y \in \mathbf{S}.$$

   Likewise, since x and y are both elements of $\mathbf{T}$ and since $\mathbf{T}$ is a subspace, we have that $x + y \in \mathbf{T}$. Therefore, since $x + y$ is in both $\mathbf{S}$ and $\mathbf{T}$, we have that

   $$x + y \in \mathbf{S} \cap \mathbf{T}.$$

   Likewise, since $x \in \mathbf{S}$ and $\mathbf{S}$ is a subspace (meaning that $\mathbf{S}$ is closed under scalar multiplication), we have that $cx \in \mathbf{S}$; similarly, $cx \in \mathbf{T}$. Therefore,

   $$cx \in \mathbf{S} \cap \mathbf{T}.$$

   Since our choices of x, y, and c were completely arbitrary, we see that $\mathbf{S} \cap \mathbf{T}$ is a subspace of $\mathbb{R}^5$.

4. Problem 2.1.22. For which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(a) Answer: Form the augmented matrix

$$\begin{bmatrix} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{bmatrix}.$$

The goal is to use elimination to get this into reduced echelon form. Subtract twice row 1 from row 2 and add row 1 to row 3 to get:

$$\begin{bmatrix} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{bmatrix}.$$

Hence, the given equation is solvable only if

$$b_2 - 2b_1 = 0 \quad \text{and} \quad b_3 + b_1 = 0.$$

In other words, the right-hand side of the equation must be a vector of the form

$$\begin{bmatrix} b_1 \\ 2b_1 \\ -b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

for any real number b_1. In other words, the column space of the given matrix is the line containing the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(b) Answer: Form the augmented matrix

$$\begin{bmatrix} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{bmatrix}.$$

Then the goal is to get this into reduced echelon form. To do so, subtract twice row 1 from row 2 and add row 1 to row 3, yielding:

$$\begin{bmatrix} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_1 \end{bmatrix}.$$

The given equation is solvable only if

$$b_3 + b_1 = 0,$$

or, equivalently, if $b_3 = -b_1$. Hence, the possible right-hand sides are vectors of the form

$$\begin{bmatrix} b_1 \\ b_2 \\ -b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

In other words, the column space of the given matrix is the plane containing the vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

5. Problem 2.1.28. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $\mathbf{C}(A)$ form a subspace.

Answer: False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, it's clear that

$$Ax = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

Hence, the column space of A consists of all vectors of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ for some real number x_1. Now, neither of the two vectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

is in the column space of A, but their sum

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is in $\mathbf{C}(A)$. Therefore, the vectors that are not in $\mathbf{C}(A)$ do are not closed under addition and so do not form a subspace.

(b) If $\mathbf{C}(A)$ contains only the zero vector, then A is the zero matrix.

Answer: True. The column space of A consists of all linear combinations of the columns of A. In particular, each column of A is an element of $\mathbf{C}(A)$. Hence, if $\mathbf{C}(A)$ contains only the zero vector, then each column of A must be the zero vector, meaning that A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

Answer: True. Suppose b is in the column space of A. That means there exists some x such that $Ax = b$. Then

$$\left(2A\right)\left(\frac{1}{2}x\right) = Ax = b,$$

so b is in the column space of $2A$. Hence, the column space of A is contained in the column space of $2A$.

On the other hand, if c is in the column space of $2A$, then there exists x such that $(2A)x = c$. But that means that

$$A(2x) = 2Ax = c,$$

so c is also in the column space of A. Hence, the column space of $2A$ is contained in the column space of A.

Since we've shown containments both directions, it must be the case that the column space of A and the column space of $2A$ are the same space.

(d) The column space of $A - I$ equals the column space of A.

Answer: False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the column space of A consists of all linear combinations of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is to say all of $\mathbb{R}^2$. On the other hand,

$$A - I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so the column space of $A - I$ consists only of the zero matrix.

6. Problem 2.2.6. Describe the attainable right-hand sides b (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

Answer: Consider the augmented matrix

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 2 & 3 & b_3 \end{bmatrix}.$$

We can convert this to reduced echelon form by subtracting twice row 1 from row 3 and subtracting 3 times row 2 from row 3:

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 - 2b_1 - 3b_2 \end{bmatrix}.$$

In order for this system to be consistent, it must be the case that

$$b_3 - 2b_1 - 3b_2 = 0$$

or, equivalently,

$$b_3 = 2b_1 + 3b_2.$$

There are no constraints on b_1 and b_2, the possible right-hand sides of the equation are vectors of the form

$$\begin{bmatrix} b_1 \\ b_2 \\ 2b_1 + 3b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

for any real numbers b_1 and b_2. In other words, the column space of A is the plane containing the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

Looking at the reduced echelon form of the matrix, we see that it is of rank 2 and that a particular solution of the given equation is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

7. Problem 2.2.20. *If A has rank r, then it has an r by r submatrix S that is invertible.* Find that submatrix S from the pivot rows and pivot columns of each A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Answer: For the first matrix, if we subtract row 1 from row 2 we get the reduced matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix},$$

so we see that the pivot columns are the first and third columns, and the pivot rows are the first and second rows. Hence, the invertible 2 by 2 submatrix of A consists of the first and third columns of the first and second rows, namely

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

For the second choice of A, subtracting row 1 from row 2 yields the reduced matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, the first column is the only pivot column and the first row is the only pivot row. Therefore, the rank of A is 1 and the invertible 1 by 1 submatrix consists of the first column of the first row, namely

$$[1].$$

For the third choice of A, we don't have to do any elimination to see that the pivot columns of A are the second and third columns and the pivot rows of A are the first and third rows.

Hence, A has rank 2 and the invertible 2 by 2 submatrix consists of the second and third columns of the first and third rows, namely

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

8. Problem 2.2.30. Execute the six steps following equation (6) to findthee column space and nullspace of A and the solution to $Ax = b$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

Answer: Form the augmented matrix $[A\ b]$:

$$\begin{bmatrix} 2 & 4 & 6 & 4 & & 4 \\ 2 & 5 & 7 & 6 & & 3 \\ 2 & 3 & 5 & 2 & & 5 \end{bmatrix}.$$

Then subtracting row 1 from rows 2 and 3 and multiplying row 1 by $\frac{1}{2}$ yields

$$\begin{bmatrix} 1 & 2 & 3 & 2 & & 2 \\ 0 & 1 & 1 & 2 & & -1 \\ 0 & -1 & -1 & -2 & & 1 \end{bmatrix}.$$

Next, subtracting twice row 2 from row 1 and adding row 2 to row 3 gives

$$\begin{bmatrix} 1 & 0 & 1 & -2 & & 4 \\ 0 & 1 & 1 & 2 & & -1 \\ 0 & 0 & 0 & 0 & & 0 \end{bmatrix}.$$

This is now in reduced echelon form, so we can answer the question. Notice that the pivot columns are the first and second columns; hence, the column space of A is the span of the first two columns of A, namely $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$. Geometrically, this is just the plane containing these two vectors.

Returning the the reduced echelon form of the augmented matrix, notice that we must have

$$\begin{aligned} x_1 &= 4 - x_3 + 2x_4 \\ x_2 &= -1 - x_3 - 2x_4, \end{aligned}$$

so the special solutions are of the form

$$x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

for some real numbers x_3 and x_4. Hence, the nullspace of A consists precisely of such linear combinations.

Finally, all solutions to the equation $Ax = b$ are of the form

$$\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

where the first term is a particular solution and the latter two terms comprise the special (or homogeneous) solutions.

9. Problem 2.2.62. Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

   Answer: The simplest way of constructing a matrix whose column space contains a given vector is to make that vector a column of the matrix. Hence, let

$$A = \begin{bmatrix} 1 & 0 & a_1 \\ 1 & 3 & a_2 \\ 5 & 1 & a_3 \end{bmatrix}.$$

   Then the column space of A automatically contains the two desired vectors, and we just need to find a_1, a_2, a_3 such that $(1, 1, 2)$ is in the nullspace. But this just means that we need to choose a_1, a_2, a_3 such that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_1 \\ 1 & 3 & a_2 \\ 5 & 1 & a_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + 2a_1 \\ 4 + 2a_2 \\ 6 + 2a_3 \end{bmatrix}.$$

   Hence, we can pick $a_1 = -1/2$, $a_2 = -2$, and $a_3 = -3$, so the matrix

$$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

   has all of the desired properties.

10. Suppose x_p is a vector in $\mathbb{R}^n$ such that

$$Ax_p = b,$$

   where A is a given $m \times n$ matrix and b is a given vector in $\mathbb{R}^m$. Prove that, if x is any solution to the equation $Ax = b$, then

$$x = x_p + x_h,$$

   where x_h is some element of the nullspace of A.

Proof. Suppose $x \in \mathbb{R}^n$ such that $Ax = b$. The goal is to find x_h such that $x = x_p + x_h$. In search of that x_h, notice that

$$A(x - x_p) = Ax - Ax_p = b - b = 0.$$

Hence, $x - x_p$ is in the nullspace of A. Letting $x_h = x - x_p$, we see that

$$x_p + x_h = x_p + (x - x_p) = x,$$

as desired.

□