Math 215 Exam #1 Practice Problem Solutions

1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

(a) If $A$ is a $2 \times 2$ matrix such that $A(x) = 0$ for all $x \in \mathbb{R}^2$, then $A$ is the zero matrix.
   **Answer:** False. If $A(x) = 0$ for all $x$, then the column space of $A$ and the nullspace of $A$ must be the same space. In particular, consider the matrix
   \[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \]
   Then, for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, we have that
   \[ Ax = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \]
   and
   \[ A(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]
   Hence, $A(x) = 0$ for all $x$, but $A \neq 0$, so $A$ gives a counterexample to the statement.

(b) A system of 3 equations in 4 unknowns can never have a unique solution.
   **Answer:** True. We can realize such an system of equations as a single matrix equation
   \[ Ax = b, \]
   where $A$ is a $3 \times 4$ matrix. Hence, $\text{rank}(A) \leq 3$, so the dimension of the nullspace of $A$ is at least 1:
   \[ \dim \text{null}(A) = 4 - \text{rank}(A) \geq 4 - 3 = 1. \]
   Hence, there must be at least one free variable in the system, meaning that, if the system is solvable at all, it must have an infinite number of solutions.

(c) If $V$ is a vector space and $S$ is a finite set of vectors in $V$, then some subset of $S$ forms a basis for $V$.
   **Answer:** False. Let $V = \mathbb{R}^2$, which is clearly a vector space, and let $S$ be the singleton set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. The single element of $S$ does not span $\mathbb{R}^2$, so no subset of $S$ can be a basis for $\mathbb{R}^2$. Hence, this provides a counterexample to the statement.

(d) Suppose $A$ is an $m \times n$ matrix such that $Ax = b$ can be solved for any choice of $b \in \mathbb{R}^m$. Then the columns of $A$ form a basis for $\mathbb{R}^m$.
   **Answer:** False. Consider the matrix
   \[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \]
   Then $A$ is already in reduced echelon form and clearly has 2 pivots, so $\text{rank}(A) = 2$. This implies that $\dim \text{col}(A) = 2$, so the column space of $A$ consists of all of $\mathbb{R}^2$. Thus, the equation $Ax = b$ can be solved for any $b \in \mathbb{R}^2$ (since any $b$ is in $\text{col}(A)$). However, the columns of $A$ are clearly not linearly independent (no set containing the zero vector can be linearly independent), so they cannot form a basis for $\mathbb{R}^2$.
   A related but true statement would be the following: “Suppose $A$ is an $m \times n$ matrix such that $Ax = b$ can be solved for any choice of $b \in \mathbb{R}^m$. Then some subset of the columns of $A$ forms a basis for $\mathbb{R}^m$.”
(e) Given 3 equations in 4 unknowns, each describes a hyperplane in $\mathbb{R}^4$. If the system of those 3 equations is consistent, then the intersection of the hyperplanes contains a line.

**Answer:** True. This is really just a restatement of (b). Translating the system of equations into a matrix equation $Ax = b$, the nullspace of $A$ must be at least one-dimensional, so the solution-space must be at least one-dimensional. Since the solution space of the matrix equation corresponds to the intersection of the hyperplanes, that intersection must be at least one-dimensional, meaning it must contain a line.

(f) If $A$ is a symmetric matrix (i.e. $A = A^T$), then $A$ is invertible.

**Answer:** False. Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

Then $A$ only has rank 1, meaning that $A$ cannot be invertible, so this gives a counterexample to the statement.

(g) If $m < n$ and $A$ is an $m \times n$ matrix such that $Ax = b$ has a solution for all $b \in \mathbb{R}^m$, then there exists $z \in \mathbb{R}^m$ such that $Ax = z$ has infinitely many solutions.

**Answer:** True. The fact that $Ax = b$ has a solution for all $b \in \mathbb{R}^m$ means that the column space of $A$ is equal to all of $\mathbb{R}^m$. Hence,

$$\text{rank}(A) = \text{dim} \text{col}(A) = m.$$ 

Since

$$\text{dim} \text{nul}(A) = n - \text{rank}(A) = n - m$$

and since $m < n$, we have that the nullspace of $A$ has some positive dimension. Since the nullspace of $A$ consists precisely of those $x \in \mathbb{R}^n$ such that $Ax = 0$, this equation has infinitely many solutions. Thus, letting $z = 0$, we see that the statement is true.

(h) The set of polynomials of degree $\leq 5$ forms a vector space.

**Answer:** True. You should check that the set of polynomials of degree $\leq 5$ satisfies all the rules for being a vector space. The important facts are this space is closed under addition and scalar multiplication.

2. For each of the following, determine whether the given subset is a subspace of the given vector space. Explain your answer.

(a) **Vector Space:** $\mathbb{R}^4$.

**Subset:** The vectors of the form

$$\begin{bmatrix} a \\ b \\ 0 \\ d \end{bmatrix}.$$ 

**Answer:** Yes, this is a subspace. If we take two vectors in the subset, say

$$\begin{bmatrix} a_1 \\ b_1 \\ 0 \\ d_1 \end{bmatrix} \text{ and } \begin{bmatrix} a_2 \\ b_2 \\ 0 \\ d_2 \end{bmatrix},$$

then their sum

$$\begin{bmatrix} a_1 \\ b_1 \\ 0 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \\ d_1 + d_2 \end{bmatrix}$$

is also in the subset, so this set is closed under addition.
Moreover, if $c \in \mathbb{R}$, then
\[
  c \begin{bmatrix}
    a_1 \\
    b_1 \\
    0 \\
    d_1
  \end{bmatrix} = \begin{bmatrix}
    ca_1 \\
    cb_1 \\
    0 \\
    cd_1
  \end{bmatrix}
\]
is in the set, so this set is closed under scalar multiplication.
Thus, the set is closed under both addition and scalar multiplication, and so is a subspace.

(b) Vector Space: $\mathbb{R}^2$.
Subset: The solutions to the equation $2x - 5y = 11$.
Answer: No, this is not a subspace. To see why, I’ll show that it is not closed under addition.
The vectors
\[
  \begin{bmatrix}
    \frac{11}{2} \\
    0
  \end{bmatrix} \text{ and } \begin{bmatrix}
    0 \\
    -\frac{11}{5}
  \end{bmatrix}
\]
are both in the set, since the pairs ($11/2, 0$) and $(0, -11/5)$ both solve the equation $2x - 5y = 11$, but
\[
  \begin{bmatrix}
    \frac{11}{2} \\
    0
  \end{bmatrix} + \begin{bmatrix}
    0 \\
    -\frac{11}{5}
  \end{bmatrix} = \begin{bmatrix}
    \frac{11}{2} \\
    \frac{1}{5}
  \end{bmatrix}
\]
is not in the set, since
\[
  2(11/2) - 5(-11/5) = 11 + 11 = 22.
\]
Therefore, the set is not closed under addition, and so is not a subspace.

(c) Vector Space: $\mathbb{R}^n$.
Subset: All $x \in \mathbb{R}^n$ such that $Ax = 2x$ where $A$ is a given $n \times n$ matrix.
Answer: Yes, this is a subspace. To prove it, suppose $x_1$ and $x_2$ are in this set, meaning that
\[
  Ax_1 = 2x_1 \quad \text{and} \quad Ax_2 = 2x_2
\]
(such vectors are called eigenvectors of $A$; we’ll learn more about them later). Then
\[
  A(x_1 + x_2) = Ax_1 + Ax_2 = 2x_1 + 2x_2 = 2(x_1 + x_2),
\]
meaning that $x_1 + x_2$ is in this set as well.
Moreover, for any $c \in \mathbb{R}$,
\[
  A(cx_1) = c(Ax_1) = c(2x_1) = 2(cx_1),
\]
so $cx_1$ is in the set as well.
Therefore, this set is closed under addition and scalar multiplication, so it is indeed a subspace.

(d) Vector Space: $\mathbb{R}^3$.
Subset: The intersection of $P_1$ and $P_2$, where $P_1$ and $P_2$ are planes through the origin.
Answer: Yes, this is a subspace. The proof is essentially the same as you gave for Problem 3(c) from HW 4.

(e) Vector Space: All polynomials.
Subset: The quadratic (i.e. degree 2) polynomials.
Answer: No, this is not a subspace. To see that it is not closed under addition, notice that if $f(t) = t^2$ and $g(t) = -t^2$, then $f$ and $g$ are both in the set of quadratic polynomials, but, since
\[
  (f + g)(t) = f(t) + g(t) = t^2 + (-t^2) = 0,
\]
the sum $f + g$ is not a quadratic polynomial.
(f) **Vector Space:** All real-valued functions.

**Subset:** Functions of the form \( f(t) = a \cos t + b \sin t + c \) for \( a, b, c \in \mathbb{R} \).

**Answer:** Yes, this is a subspace. If \( a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R} \) and I define

\[
  f(t) = a_1 \cos t + b_1 \sin t + c_1
\]

and

\[
  g(t) = a_2 \cos t + b_2 \sin t + c_2,
\]

then \( f \) and \( g \) are in the given subset. The sum has the form

\[
  f(t) + g(t) = (a_1 + a_2) \cos t + (b_1 + b_2) \sin t + (c_1 + c_2),
\]

so \( f + g \) is also in the subset, which is, therefore, closed under addition. Also, if \( r \in \mathbb{R} \), then

\[
  rf(t) = r(a_1 \cos t + b_1 \sin t + c_1) = (ra_1) \cos t + (rb_1) \sin t + (rc_1),
\]

so \( rf \) is in the subset, which is, therefore, closed under scalar multiplication. Hence, we can conclude that this subset is actually a subspace.

3. Consider the matrix

\[
  A = \begin{bmatrix}
  1 & a \\
  a & 1
  \end{bmatrix}.
\]

(a) Under what conditions on \( a \) is \( A \) invertible?

**Answer:** The matrix \( A \) is invertible if and only if it has rank 2. To see what the rank is, we do elimination. The first step is to subtract \( a \) times row 1 from row 2, yielding

\[
  \begin{bmatrix}
  1 & a \\
  0 & 1 - a^2
  \end{bmatrix}.
\]

Then this has a second pivot if and only if \( 1 - a^2 \neq 0 \), meaning that \( a^2 \neq 1 \), or \( a \neq \pm 1 \). Thus, \( A \) is invertible so long as \( a \) is neither 1 nor \(-1 \).

(b) Choose a non-zero value of \( a \) that makes \( A \) invertible and determine \( A^{-1} \).

**Answer:** Choose \( a = 2 \). Recall that we can find the inverse of \( A \) by converting the left side of the following augmented matrix to the identity:

\[
  \begin{bmatrix}
  1 & 2 & 1 & 0 \\
  2 & 1 & 0 & 1
  \end{bmatrix}.
\]

Subtract twice row 1 from row 2:

\[
  \begin{bmatrix}
  1 & 2 & 1 & 0 \\
  0 & -3 & -2 & 1
  \end{bmatrix}.
\]

Scale the second row by \(-\frac{1}{3} \) and also subtract twice the result from row 1:

\[
  \begin{bmatrix}
  1 & 0 & -\frac{1}{3} & 0 \\
  0 & 1 & \frac{2}{3} & -\frac{1}{3}
  \end{bmatrix}.
\]

Therefore,

\[
  A^{-1} = \begin{bmatrix}
  -\frac{1}{3} & \frac{2}{3} \\
  \frac{2}{3} & -\frac{1}{3}
  \end{bmatrix}.
\]
(c) For each value of $a$ that makes $A$ non-invertible, determine the dimension of the nullspace of $A$.

**Answer:** When $a = 1$, the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which, after subtracting row 1 from row 2, reduces to

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$ 

Hence, $A$ has rank 1, so the nullspace has dimension

$$\dim \text{nul}(A) = 2 - \text{rank}(A) = 2 - 1 = 1.$$  

4. Consider the system of equations

$$x_1 + 2x_2 + x_3 - 3x_4 = b_1$$

$$x_1 + 2x_2 + 2x_3 - 5x_4 = b_2,$$

$$2x_1 + 4x_2 + 3x_3 - 8x_4 = b_3.$$ 

(a) Find all solutions when the above system is homogeneous (i.e. $b_1 = b_2 = b_3 = 0$). Find a basis for the space of solutions to the homogeneous system.

**Answer:** Convert the system into the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 \\ 1 & 2 & 2 & -5 & 0 \\ 2 & 4 & 3 & -8 & 0 \end{bmatrix}.$$ 

Now do elimination to get the reduced echelon form. First, subtract row 1 from row 2 and subtract twice row 1 from row 3:

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}.$$ 

Now, subtract row 2 from both row 1 and row 3:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Then this system is consistent provided that

$$x_1 = -2x_2 + x_4$$

$$x_3 = 2x_4.$$ 

Hence, the solutions to the homogeneous equation are those vectors of the form

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$ 

for $x_2, x_4 \in \mathbb{R}$. Then a basis for the space of solutions to the homogeneous system (i.e. nullspace of the corresponding matrix) is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$ 

4. Consider the system of equations

$$x_1 + 2x_2 + x_3 - 3x_4 = b_1$$

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**Answer:** Convert the system into the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 \\ 1 & 2 & 2 & -5 & 0 \\ 2 & 4 & 3 & -8 & 0 \end{bmatrix}.$$ 

Now do elimination to get the reduced echelon form. First, subtract row 1 from row 2 and subtract twice row 1 from row 3:

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}.$$ 

Now, subtract row 2 from both row 1 and row 3:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Then this system is consistent provided that

$$x_1 = -2x_2 + x_4$$

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Hence, the solutions to the homogeneous equation are those vectors of the form

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$ 

for $x_2, x_4 \in \mathbb{R}$. Then a basis for the space of solutions to the homogeneous system (i.e. nullspace of the corresponding matrix) is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$ 

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(b) Let $S$ be the set of vectors $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that the system can be solved. What is the dimension of $S$?

**Answer:** Letting $A$ be the matrix of the system, we know that the set of vectors $b$ for which the system can be solved is the column space of $A$. Since $A$ is $3 \times 4$, we know that

$$\text{rank}(A) + \text{dim nul}(A) = 4.$$  

Since, from part (a), we know that the dimension of the nullspace is 2, this implies that the column space of $A$ is two-dimensional.

(c) It’s easy to check that the vector $v = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ is a solution to the system that arises when $b_1 = 3$, $b_2 = 5$, and $b_3 = 8$. Find all the solutions to this system.

**Answer:** All solutions $x$ to the system $Ax = b$ take the form $x = x_p + x_h$, where $x_p$ is a particular solution and $x_h$ is the homogeneous solution to the corresponding homogeneous problem. Thus, we can let $x_p = v$, which we’re told solves the system and we see that, using part (a), the general solution is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

where $x_2, x_4 \in \mathbb{R}$.  

\[ \text{6} \]