Math 115 Exam #1 Practice Problems

For each of the following, say whether it converges or diverges and explain why.

1. $\sum_{n=1}^{\infty} \frac{n^3}{n^5+3}$

   Answer: Notice that
$$\frac{n^3}{n^5+3} < \frac{n^3}{n^5} = \frac{1}{n^2}$$
   for all n. Therefore, since $\sum \frac{1}{n^2}$ converges (it's a p-series with $p = 2 > 1$), the series $\sum \frac{n^3}{n^5+3}$ also converges by the comparison test.

2. $\sum_{n=1}^{\infty} \frac{3^n}{4^n+4}$

   Answer: Notice that
$$\frac{3^n}{4^n+4} < \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$$
   for all n. Therefore, since $\sum \left(\frac{3}{4}\right)^n$ converges (it's a geometric series with $r = \frac{3}{4} < 1$), the series $\sum \frac{3^n}{4^n+4}$ also converges by the comparison test.

3. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

   Answer: Using the Root Test:
$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{2^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}.$$
   Since the limit is less than 1, the Root Test says that the series converges absolutely.

4. For what values of p does the series $\sum_{n=1}^{\infty} \frac{n^p}{2+n^3}$ converge?

   Answer: Doing a limit comparison to $\sum \frac{1}{n^{3-p}}$, I see that
$$\lim_{n\to\infty} \frac{\frac{n^p}{2+n^3}}{\frac{1}{n^{3-p}}} = \lim_{n\to\infty} \frac{n^3}{2+n^3} = \frac{1}{2}.$$
   Therefore, the series converges if and only if the series $\sum \frac{1}{n^{3-p}}$ converges. This happens when $3-p > 1$, which is to say when $p < 2$. So the given series converges when $p < 2$.

5. We would like to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4+3}$ by using the sum of the first ten terms. Of course, the exact error is the sum of all the terms from the 11th on, i.e., $\sum_{n=11}^{\infty} \frac{1}{n^4+3}$. Show that this error is less than 1/3000 by comparing this with the sum of $1/n^4$ and then by estimating this latter sum using an appropriate integral.

   Answer: Notice that
$$\frac{1}{n^4+3} < \frac{1}{n^4}$$
   for all n, so
$$\sum_{n=11}^{\infty} \frac{1}{n^4+3} < \sum_{n=11}^{\infty} \frac{1}{n^4}.$$
   In turn, the sum on the right is less than
$$\int_{10}^{\infty} \frac{1}{x^4}\,dx = \left.\frac{-1}{3x^3}\right]_{10}^{\infty} = \frac{1}{3000},$$
   so we see that the error is less than 1/3000.

6. Does the series

$$\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}$$

converge or diverge?

Answer: Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!(n+2)!}{(3n+3)!}}{\frac{n!(n+1)!}{(3n)!}} \right| = \lim_{n\to\infty} \frac{(n+1)!(n+2)!}{(3n+3)!} \cdot \frac{(3n)!}{n!(n+1)!} = \lim_{n\to\infty} \frac{(n+1)(n+2)}{(3n+3)(3n+2)(3n+1)}$$

$$= \lim_{n\to\infty} \frac{n^2+3n+2}{27n^3+54n^2+33n+6}.$$

Dividing numerator and denominator by n^3 yields

$$\lim_{n\to\infty} \frac{\frac{1}{n}+\frac{3}{n^2}+\frac{2}{n^3}}{27+\frac{54}{n}+\frac{33}{n^2}+\frac{6}{n^3}} = 0.$$

Since $0 < 1$, the Ratio Test says that the series converges absolutely.

7. Does the series

$$\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$$

converge absolutely, converge conditionally, or diverge?

Answer: Notice that

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos\left(\lim_{x\to\infty} \frac{1}{x}\right) = \cos(0) = 1$$

since cosine is a continuous function. Therefore, the terms

$$(-1)^n \cos\left(\frac{1}{n}\right)$$

are not going to zero, so the Divergence Test says that the series diverges.

8. Determine the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n^3 x^{3n}}{n^4+1}$$

Answer: Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3 x^{3n+3}}{(n+1)^4+1}}{\frac{n^3 x^3 n}{n^4+1}} \right| = \lim_{n\to\infty} \left| \frac{(n^4+1)(n+1)^3 x^3}{n^3((n+1)^4+1)} \right|$$

$$= |x|^3 \lim_{n\to\infty} \frac{(n^4+1)(n+1)^3}{n^3((n+1)^4+1)}$$

$$= |x|^3 \lim_{n\to\infty} \frac{n^7+\dots}{n^7+\dots}$$

$$= |x|^3$$

which is less than 1 when $|x| < 1$, so the radius of convergence is 1.

9. Consider the sequence defined by $a_n = \frac{(-1)^n+n}{(-1)^n-n}$. Does this sequence converge and, if it does, to what limit?

Answer: Dividing numerator and denominator by n, we have that

$$\lim_{n\to\infty} \frac{(-1)^n+n}{(-1)^n-n} = \lim_{n\to\infty} \frac{\frac{1}{n}\left((-1)^n+n\right)}{\frac{1}{n}\left((-1)^n-n\right)} = \lim_{n\to\infty} \frac{\frac{(-1)^n}{n}+1}{\frac{(-1)^n}{n}-1} = \frac{1}{-1} = -1,$$

so the sequence converges to -1.

10. Find the value of the series

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^{n-1}}.$$

Answer: I can re-write the terms as:

$$\frac{1+2^n}{3^{n-1}} = \frac{1}{3^{n-1}} + \frac{2^n}{3^{n-1}} = \left(\frac{1}{3}\right)^{n-1} + 2\left(\frac{2}{3}\right)^{n-1}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}.$$

Shifting the indices of the sums down by one yields

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n} + \sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^{n}.$$

These are both geometric series, so I can sum them using the formula for geometric series:

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n} + \sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^{n} = \frac{1}{1-\frac{1}{3}} + \frac{2}{1-\frac{2}{3}} = \frac{3}{2} + 6 = \frac{15}{2}.$$

11. Does the series

$$\sum_{n=1}^{\infty} \frac{n+5}{n\sqrt{n+3}}$$

converge or diverge?

Answer: Do a limit comparison to $\sum \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{\frac{n+5}{n\sqrt{n+3}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{(n+5)\sqrt{n}}{n\sqrt{n+3}} = \lim_{n\to\infty} \frac{n^{3/2}+5n^{1/2}}{\sqrt{n^3+3n^2}}.$$

Dividing numerator and denominator by $n^{3/2}$ yields

$$\lim_{n\to\infty} \frac{\frac{1}{n^{3/2}}\left(n^{3/2}+5n^{1/2}\right)}{\frac{1}{n^{3/2}}\sqrt{n^3+3n^2}} = \lim_{n\to\infty} \frac{1+\frac{5}{n}}{\sqrt{\frac{1}{n^3}\left(n^3+3n^2\right)}} = \lim_{n\to\infty} \frac{1+\frac{5}{n}}{\sqrt{1+\frac{3}{n}}} = 1.$$

Therefore, since $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ diverges (it's a p-series with $p = 1/2 < 1$), the Limit Comparison Test says that the given series also diverges.

12. Does the series

$$\sum_{n=1}^{\infty} \frac{3+\cos n}{e^n}$$

converge or diverge?

Answer: Notice that

$$|3+\cos n| \leq 4$$

for all n, so

$$\left|\frac{3+\cos n}{e^n}\right| = \frac{|3+\cos n|}{e^n} \leq \frac{4}{e^n} = 4\left(\frac{1}{e}\right)^n$$

for all n. Since $\frac{1}{e} < 1$, the series $\sum 4\left(\frac{1}{e}\right)^n$ converges and so, by the comparison test, $\sum \left|\frac{3+\cos n}{e^n}\right|$ also converges.

Hence, the series $\sum \frac{3+\cos n}{e^n}$ converges absolutely.

13. Does the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n^2+1}}$$

converge absolutely, converge conditionally, or diverge?

Answer: The terms $\frac{1}{\sqrt{n^2+1}}$ are decreasing and go to zero (you should check this), so the Alternating Series Test says that the series converges.

To see that the series does not converge absolutely, it suffices to show that the series

$$\sum_{n=0}^{\infty} \left|(-1)^n \frac{1}{\sqrt{n^2+1}}\right| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

diverges. To see this, do a limit comparison with the divergent series $\sum \frac{1}{n}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} &= \lim_{n\to\infty} \frac{\frac{1}{n}n}{\frac{1}{n}\sqrt{n^2+1}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n^2}\left(n^2+1\right)}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} \\
&= 1.
\end{aligned}$$

Since the limit is finite and non-zero, the limit comparison test says that the series $\sum \frac{1}{\sqrt{n^2+1}}$ diverges.

14. Does the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{\pi^n}$$

converge absolutely, converge conditionally, or diverge?

Answer: Using the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}\frac{(n+1)!}{\pi^{n+1}}}{(-1)^n \frac{n!}{\pi^n}}\right| = \lim_{n\to\infty} \frac{n+1}{\pi} = \infty.$$

Therefore, the Ratio Test says that the series diverges.