MATH 104 HW 6

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§6.5

4. The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 h, how much raw sugar will remain after another 14 h?

Answer: Let y denote the amount of raw sugar remaining in the processing. Since the rate of change is proportional to the amount remaining, we know that $\frac{dy}{dt} = ky$ for some constant k. Therefore, as we saw in class, $y = y_0e^{kt}$ where t is measured in hours. y_0 is the initial amount of sugar, so, in this case, $y_0 = 1000$. After 10 hours, 800 kg remains, so

$$800 = 1000e^{k(10)}$$

Dividing by 1000, we see that

$$0.8 = e^{10k}.$$

Taking the natural logarithm of both sides,

$$\ln 0.8 = 10k,$$

so $k = \frac{\ln 0.8}{10}$. Therefore,

$$y(24) = 1000e^{\frac{\ln 0.8}{10}(24)} \approx 585kg$$

of raw sugar remain after 24 hours.

8. A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 h there are 10,000 bacteria. At the end of 5 h there are 40,000. How many bacteria were present initially?

Answer: Let y denote the number of bacteria present. Since the population increases exponentially, $y = y_0e^{kt}$ where t is measured in hours. Now,

$$10,000 = y(3) = y_0e^{k(3)} = y_0e^{3k},$$

so $y_0 = 10,000e^{-3k}$. Also, we know that

$$40,000 = y(5) = y_0e^{k(5)} = \left(10,000e^{-3k}\right)e^{5k} = 10,000e^{5k-3k} = 10,000e^{2k}.$$

Dividing both sides by $10,000$, we see that

$$4 = e^{2k}.$$

Taking the natural logarithm of both sides,

$$\ln 4 = 2k,$$

so $k = \frac{\ln 4}{2}$. Therefore,

$$y_0 = 10,000e^{-3k} = 10,000e^{-3\frac{\ln 4}{2}} = 1250.$$

Therefore, the experiment started with 1250 bacteria.

11. Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

Answer: If y denotes the amount of oil, then we can model the output of the well by the equation

$$y = y_0 e^{-0.1t}.$$

When the well's output has fallen to one-fifth of its current output, then we will have that

$$\frac{1}{5}y_0 = y = y_0 e^{-0.1t}.$$

Therefore, dividing by y_0,

$$\frac{1}{5} = e^{-0.1t}.$$

Taking the natural log of both sides,

$$\ln \frac{1}{5} = -0.1t.$$

Therefore,

$$t = \frac{\ln \frac{1}{5}}{-0.1} = \frac{-\ln 5}{-0.1} = \frac{\ln 5}{0.1} = 10 \ln 5 \approx 16.09.$$

Therefore, we can expect the well's output to fall to one-fifth its present value in a little over 16 years.

13. You have just placed A_0 dollars in a bank account that pays 4%, compounded continuously.

(a): How much money will you have in the account in 5 years?
(b): How long will it take your money to double? to triple?

Answer:

(a): Since the interest is compounded continuously, we can model this situation by

$$A(t) = A_0 e^{0.04t}$$

where A denotes the amount of money in your account. Hence, 5 years down the road, you will have

$$A(5) = A_0 e^{0.04(5)} = A_0 e^{0.2} \approx 1.22A_0$$

dollars in your account.

(b): Your money has doubled when $A(t) = 2A_0$. Hence, we write

$$2A_0 = A(t) = A_0 e^{0.04t}$$

and solve for t. Dividing by A_0 yields

$$2 = e^{0.04t}$$

so

$$\ln 2 = 0.04t.$$

Hence,

$$t = \frac{\ln 2}{0.04} \approx 17.33.$$

Hence, your money will double in 17 years and 4 months.

Your money has tripled when

$$3A_0 = A(t) = A_0 e^{0.04t}.$$

Hence,

$$3 = e^{0.04t}$$

and so

$$\ln 3 = 0.04t.$$

Therefore, your money will triple in

$$t = \frac{\ln 3}{0.04} \approx 26.46 \text{ years},$$

or about 26 years and 6 months.

15. [...] What rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90?

Answer: Since continuously compounded interest is modeled by $A(t) = Pe^{rt}$ and the original principle $P = 1000$ pounds, we want to solve the equation

$$90,000 = 1000e^{r(100)}$$

for r. Dividing both sides by 1000, we see that

$$90 = e^{100r}.$$

Taking the natural logarithm,

$$\ln 90 = 100r,$$

so

$$r = \frac{\ln 90}{100} \approx 0.045,$$

or 4.5% annual interest.

§6.6

6. Find
$$\lim_{x\to\infty} \frac{x-8x^2}{12x^2+5x}.$$

Answer: Since $x-8x^2 \to -\infty$ as $x \to \infty$ and $12x^2+5x \to \infty$ as $x \to \infty$, we can use L'Hôpital's rule to see that
$$\lim_{x\to\infty} \frac{x-8x^2}{12x^2+5x} = \lim_{x\to\infty} \frac{1-16x}{24x+5}.$$
Now, as $x \to \infty$, $1-16x \to -\infty$ and $24x+5 \to \infty$, so we can use L'Hôpital's rule again to see that
$$\lim_{x\to\infty} \frac{x-8x^2}{12x^2+5x} = \lim_{x\to\infty} \frac{1-16x}{24x+5} = \lim_{x\to\infty} \frac{-16}{24} = \frac{-2}{3}.$$

9. Find
$$\lim_{x\to 0} \frac{8x^2}{\cos x - 1}.$$

Answer: Since $8(0)^2 = 0$ and $\cos 0 - 1 = 0$, we can use L'Hôpital's Rule to see that
$$\lim_{x\to 0} \frac{8x^2}{\cos x - 1} = \lim_{x\to 0} \frac{16x}{-\sin x}.$$
Now, $16(0) = 0$ and $-\sin 0 = 0$, so we use L'Hôpital's Rule again to see that
$$\lim_{x\to 0} \frac{16x}{-\sin x} = \lim_{x\to 0} \frac{16}{-\cos x} = \frac{16}{-1} = -16.$$

16. Find
$$\lim_{x\to\pi/2} \frac{\ln(\csc x)}{(x-\pi/2)^2}.$$

Answer: Since $\csc \pi/2 = \frac{1}{\sin \pi/2} = \frac{1}{1} = 1$ and $\ln 1 = 0$, we see that the numerator goes to zero. The denominator also goes to zero, so we use L'Hôpital's Rule to see that
$$\lim_{x\to\pi/2} \frac{\ln(\csc x)}{(x-\pi/2)^2} = \lim_{x\to\pi/2} \frac{\frac{1}{\csc x}\cdot -\csc x \cot x}{2(x-\pi/2)} = \lim_{x\to\pi/2} \frac{-\cot x}{2(x-\pi/2)}.$$
Now, $\cot \pi/2 = 0$ and $2(\pi/2 - \pi/2) = 0$, so we use L'Hôpital's Rule again to see that
$$\lim_{x\to\pi/2} \frac{-\cot x}{2(x-\pi/2)} = \lim_{x\to\pi/2} \frac{\csc^2 x}{2} = \frac{1}{2}.$$

26. Find
$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)}.$$

Answer: Since both top and bottom tend to infinity as $x \to \infty$, we can use L'Hôpital's Rule to see that
$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)} = \lim_{x\to\infty} \frac{\frac{1}{\ln 2}\cdot\frac{1}{x}}{\frac{1}{\ln 3}\cdot\frac{1}{x+3}} = \lim_{x\to\infty} \frac{\ln 3}{\ln 2}\cdot\frac{x+3}{x}.$$

Now, as $x \to \infty$, both x and $x+3$ tend to infinity, so

$$\lim_{x\to\infty} \frac{x+3}{x} = \lim_{x\to\infty} \frac{1}{1} = 1,$$

so we see that

$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)} = \lim_{x\to\infty} \frac{\ln 3}{\ln 2} \cdot \frac{x+3}{x} = \frac{\ln 3}{\ln 2}.$$

56. Find the following limit without using L'Hôpital's Rule (which doesn't help):

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x}.$$

Answer: Re-write and compute:

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \cos x = 1.$$

59. Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

(a):

$$\lim_{x\to 0^+} x \ln x = 0 \cdot (-\infty) = 0.$$

(b):

$$\lim_{x\to 0^+} x \ln x = 0 \cdot (-\infty) = -\infty.$$

(c):

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \frac{-\infty}{\infty} = -1.$$

(d):

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

Answer: Since $\ln x \to -\infty$ and $1/x \to \infty$ as $x \to 0^+$, (d) is a proper application of L'Hôpital's Rule, so it is the correct calculation of the limit. The others lead to wrong answers because infinite arithmetic doesn't act like normal arithmetic, so we can't just do the arithmetic tricks suggested in (a), (b) and (c).

§6.7

2. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate? Which grow slower?

Answer:

(a): $10x^4 + 30x + 1$: We compare:

$$\lim_{x\to\infty} \frac{10x^4 + 30x + 1}{e^x}.$$

Now, by 4 straight applications of L'Hôpital's Rule, we see that

$$\begin{aligned}\lim_{x\to\infty}\frac{10x^4+30x+1}{e^x} &= \lim_{x\to\infty}\frac{40x^3+30}{e^x}\\ &= \lim_{x\to\infty}\frac{120x^2}{e^x}\\ &= \lim_{x\to\infty}\frac{240x}{e^x}\\ &= \lim_{x\to\infty}\frac{240}{e^x}\\ &= 0,\end{aligned}$$

so $10x^4+30x+1$ grows slower than e^x.

(b): $x\ln x - x$: We compare:

$$\begin{aligned}\lim_{x\to\infty}\frac{x\ln x - x}{e^x} &= \lim_{x\to\infty}\frac{x(1/x)+\ln x - 1}{e^x}\\ &= \lim_{x\to\infty}\frac{\ln x}{e^x}\\ &= \lim_{x\to\infty}\frac{\frac{1}{x}}{e^x}\\ &= \lim_{x\to\infty}\frac{1}{xe^x}\\ &= 0\end{aligned}$$

by two applications of L'Hôpital's Rule, so $x\ln x - x$ grows slower than e^x.

(c): $\sqrt{1+x^4}$ grows at the same rate as x^2 by a computation similar to that done in problem 18. Now, x^2 grows slower than e^x as we showed in class, so $\sqrt{1+x^4}$ grows slower than e^x.

(d): $(5/2)^x$:

$$\lim_{x\to\infty}\frac{(5/2)^x}{e^x} = \lim_{x\to\infty}\left(\frac{5/2}{e}\right)^x = 0$$

since $5/2 < e$. Hence, $(5/2)^x$ grows slower than e^x.

(e): e^{-x} does grow at all; it converges to 0 as $x\to\infty$. Hence, it certainly grows slower than e^x.

(f): xe^x:

$$\lim_{x\to\infty}\frac{xe^x}{e^x} = \lim_{x\to\infty} x = \infty,$$

so xe^x grows faster than e^x.

(g): $e^{\cos x}$: Since $|\cos x| \le 1$ for all x, $e^{\cos x} \le e^1 = e$ for all x. Hence,

$$\lim_{x\to\infty}\frac{e^{\cos x}}{e^x} \le \lim_{x\to\infty}\frac{e}{e^x} = 0,$$

so $e^{\cos x}$ grows slower than e^x.

(h): e^{x-1}:

$$\lim_{x\to\infty} \frac{e^{x-1}}{e^x} = \lim_{x\to\infty} e^{x-1}e^{-x} = \lim_{x\to\infty} e^{-1} = \frac{1}{e},$$

so e^{x-1} grows at the same rate as e^x.

5. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

Answer:

(a): $\log_3 x$:

$$\lim_{x\to\infty} \frac{\log_3 x}{\ln x} = \lim_{x\to\infty} \frac{\frac{\ln x}{\ln 3}}{\ln x} = \frac{1}{\ln 3},$$

so $\log_3 x$ grows at the same rate as $\ln x$.

(b): $\ln 2x$:

$$\lim_{x\to\infty} \frac{\ln 2x}{\ln x} = \lim_{x\to\infty} \frac{\ln 2 + \ln x}{\ln x} = \lim_{x\to\infty} \frac{\ln 2}{\ln x} + 1 = 0 + 1 = 1,$$

so $\ln 2x$ grows at the same rate as $\ln x$.

(c): $\ln \sqrt{x}$: By L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{\ln \sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{2x}}{\frac{1}{x}} = \frac{1}{2},$$

so $\ln \sqrt{x}$ and $\ln x$ grow at the same rate.

(d): $\sqrt{x}$: By L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{2} = \infty,$$

so $\sqrt{x}$ grows faster than $\ln x$.

(e): x: By L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty,$$

so x grows faster than $\ln x$.

(f): $5\ln x$:

$$\lim_{x\to\infty} \frac{5\ln x}{\ln x} = \lim_{x\to\infty} 5 = 5,$$

so $5\ln x$ and $\ln x$ grow at the same rate.

(g): $1/x$ tends to 0 as $x \to \infty$, so certainly $1/x$ grows slower than $\ln x$.

(h): e^x grows faster than x, as we've already seen, and we showed in (e) above that x grows faster than $\ln x$, so certainly e^x grows faster than $\ln x$.

10. True, or false? As $x \to \infty$,

Answer:

(a): $\frac{1}{x+3} = O\left(\frac{1}{x}\right)$
Consider
$$\frac{\frac{1}{x+3}}{\frac{1}{x}} = \frac{x}{x+3} < 1$$
for all $x > 0$, so, indeed, $\frac{1}{x+3} = O\left(\frac{1}{x}\right)$.

(b): $\frac{1}{x} + \frac{1}{x^2} = O\left(\frac{1}{x}\right)$
Consider
$$\frac{\frac{1}{x} + \frac{1}{x^2}}{\frac{1}{x}} = 1 + \frac{1}{x}.$$
Now, for $x > 1$, $1 + \frac{1}{x} \leq 2$, so we see that $\frac{1}{x} + \frac{1}{x^2} = O\left(\frac{1}{x}\right)$.

(c): $\frac{1}{x} - \frac{1}{x^2} = o\left(\frac{1}{x}\right)$
We check
$$\lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{\frac{1}{x}} = \lim_{x\to\infty} 1 - \frac{1}{x} = 1,$$
so $\frac{1}{x} - \frac{1}{x^2} \neq o\left(\frac{1}{x}\right)$.

(d): $2 + \cos x = O(2)$
Check
$$\frac{2+\cos x}{2} = 1 + \frac{\cos x}{2}.$$
Since $|\cos x| \leq 1$, $\frac{\cos x}{2} \leq \frac{1}{2}$, so
$$1 + \frac{\cos x}{2} \leq \frac{3}{2}$$
so, indeed, $2 + \cos x = O(2)$.

(e): $e^x + x = O(e^x)$.
Check
$$\frac{e^x + x}{e^x} = 1 + \frac{x}{e^x}.$$
Now, for all $x \geq 0$, $x < e^x$, so $\frac{x}{e^x} \leq 1$. Hence,
$$1 + \frac{x}{e^x} \leq 2,$$
so $e^x + x = O(e^x)$.

(f): $x \ln x = o(x^2)$
Compute the limit
$$\lim_{x\to\infty} \frac{x \ln x}{x^2} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$
by L'Hôpital's Rule, so we see that $x \ln x = o(x^2)$.

(g): $\ln(\ln x) = O(\ln x)$
We try to check
$$\frac{\ln(\ln x)}{\ln x}.$$

Now, since $x < e^x$ for all $x > 0$, it must be the case that $x > \ln x$ for all $x > 0$. Hence, since $\ln x$ is an increasing function,

$$\frac{\ln(\ln x)}{\ln x} < \frac{\ln x}{\ln x} = 1,$$

so we see that $\ln(\ln x) = O(\ln x)$.

(h): $\ln x = o(\ln(x^2+1))$

Compute the limit:

$$\lim_{x\to\infty} \frac{\ln x}{\ln(x^2+1)} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x^2+1}\cdot 2x} = \lim_{x\to\infty} \frac{x^2+1}{2x^2}.$$

Now, by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x^2+1}{2x^2} = \lim_{x\to\infty} \frac{2x}{4x} = \frac{1}{2}.$$

So $\ln x$ and $\ln(x^2+1)$ grow at the same rate, so $\ln x \neq o(\ln(x^2+1))$.

18. Show that $\sqrt{x^4+x}$ and $\sqrt{x^4-x^3}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as x^2 as $x \to \infty$.

Answer: Comparing $\sqrt{x^4+x}$ to x^2, we see that

$$\lim_{x\to\infty} \frac{\sqrt{x^4+x}}{x^2} = \lim_{x\to\infty} \sqrt{\frac{x^4+x}{x^4}} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x^3}} = 1,$$

so $\sqrt{x^4+x}$ and x^2 grow at the same rate.

Similarly,

$$\lim_{x\to\infty} \frac{\sqrt{x^4-x^3}}{x^2} = \lim_{x\to\infty} \sqrt{\frac{x^4-x^3}{x^4}} = \lim_{x\to\infty} \sqrt{1-\frac{1}{x}} = 1,$$

so $\sqrt{x^4-x^3}$ grows at the same rate as x^2 and, therefore, at the same rate as $\sqrt{x^4+x}$.

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