Homotopy and link homotopy

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Joint work with Frederick R. Cohen and Rafal Komendarczyk
Consider a *parametrized* $n$-component link $L_1, L_2, L_3, \ldots, L_n$, where

$$L_i : S^1 \to \mathbb{R}^3$$

have disjoint images.
Given an $n$-component link $L = \{L_1, \ldots, L_n\}$, there is a natural evaluation map

$$F_L : S^1 \times \ldots \times S^1 \longrightarrow \text{Conf}(n)$$

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The Evaluation Map

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Here $\text{Conf}(n)$ is the configuration space of $n$ distinct points in $\mathbb{R}^3$:

$$\text{Conf}(n) = \text{Conf}_n \mathbb{R}^3 = \{(x_1, \ldots, x_n) \in (\mathbb{R}^3)^n : x_i \neq x_j \text{ for } i \neq j\}.$$
Definition

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The $\kappa$ invariant

$$F_L : (S^1)^n \longrightarrow \text{Conf}(n).$$

Link homotopies of $L$ induce homotopies of $F_L$.

We get an induced map

$$\kappa : \text{Link}(n) \longrightarrow [(S^1)^n, \text{Conf}(n)]$$

$$L \quad \mapsto \quad [F_L]$$
Theorem (Gauss)

The degree of the composition is equal to the linking number.
\[ S^1 \times S^1 \xrightarrow{(t_1, t_2)} (L_1(t_1), L_2(t_2)) \xrightarrow{\frac{L_1(t_1) - L_2(t_2)}{|L_1(t_1) - L_2(t_2)|}} S^2 \]

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Since the linking number classifies 2-component links up to link homotopy and the degree classifies maps \( S^1 \times S^1 \to S^2 \) up to homotopy, \( \kappa \) is bijective for \( n = 2 \).
\[ S^1 \times S^1 \times S^1 \hookrightarrow \text{Conf}_3 S^3 \xrightarrow{\text{hom. equiv.}} S^3 \times S^2 \xrightarrow{\pi} S^2. \]
\[ S^1 \times S^1 \times S^1 \subset \text{Conf}_3 S^3 \xrightarrow{\text{hom. equiv.}} S^3 \times S^2 \xrightarrow{\pi} S^2. \]

**Theorem (with DeTurck, Gluck, Komendarczyk, Melvin, and Vela-Vick)**

The homotopy periods of the above map give the complete set of link homotopy invariants of \(3\)-component links.

**Corollary**

\[ \kappa : \text{Link}(3) \rightarrow \left[ (S^1)^3, \text{Conf}(3) \right] \] is injective (though not bijective).
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Conjecture (Koschorke)

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Let $\text{BLink}(n)$ denote the set of homotopy Brunnian $n$-component links, meaning every $(n-1)$-component sublink is link homotopically trivial.
Koschorke’s Conjecture

**Conjecture (Koschorke)**

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**Theorem (Koschorke)**

The restriction \( \kappa : \text{BLink}(n) \to [(S^1)^n, \text{Conf}(n)] \) is injective.
Let $\text{CLink}(n)$ be the set of $n$-component links with some $(n - 1)$-component sublink which is link homotopically trivial.
Main Theorem

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Theorem (with Cohen and Komendarczyk)

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Theorem (Habegger–Lin) The map $H(n) \to \text{Link}(n)$ induced by the Markov closure is surjective. Moreover, if two string links close up to link-homotopic links, then they are related by a sequence of conjugations and "partial conjugations."
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For each $i = 1, \ldots, n$, there is a natural projection

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$$C\mathcal{H}(n) = \bigcup_i \ker \delta_i$$

is the set of string links with a trivial $(n - 1)$-stranded sublink.
Key Proposition 1

$\mathcal{B}\mathcal{H}(n)$ is the center of $\mathcal{H}(n)$. 
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Corollary

The restriction of the Markov closure map $B\mathcal{H}(n) \to B\text{Link}(n)$ is bijective.
Torus Homotopy Groups

Let $T(n)$ be the $n$th Fox torus homotopy group of $\text{Conf}(n)$:

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If $L \in \text{BLink}(n)$, then $L - L_n$ is link homotopically trivial, so the restriction of $F_L$ to the face $(S^1)^{n-1} \times \{\ast\}$ is homotopic to the constant map.
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Therefore, we can interpret $\kappa(L) = [F_L]$ as an element of $T(n)$. 
**Key Proposition 2**

*The diagram*

\[
\begin{array}{ccc}
B\mathcal{H}(n) & \xrightarrow{\phi} & T(n) \\
\downarrow & & \downarrow p^# \\
\text{BLink}(n) & \xrightarrow{\kappa} & [(S^1)^n, \text{Conf}(n)]
\end{array}
\]

commutes and \(p^# \circ \phi\) is injective. Therefore \(\kappa\) is injective as well.
Thanks!