Minimal Surfaces

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Soap Films

A soap film seeks to minimize its surface energy, which is proportional to area. Hence, a soap film achieves a minimum area among all surfaces with the same boundary.
Computer-generated soap film
Double catenoid
Costa-Hoffman-Meeks Surface
Quick Background on Surfaces

Recall that, if $S \subset \mathbb{R}^3$ is a regular surface, the inner product $\langle \ , \ \rangle_p$ on $T_p S$ is a symmetric bilinear form. The associated quadratic form

$$I_p : T_p S \to \mathbb{R}$$

given by

$$I_p(W) = \langle W, W \rangle_p$$

is called the first fundamental form of $S$ at $p$. 
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If \( X : U \subset \mathbb{R}^2 \to V \subset S \) is a local parametrization of \( S \) with coordinates \((u, v)\) on \( U \), then the vectors \( X_u \) and \( X_v \) form a basis for \( T_pS \) with \( p \in V \). The first fundamental form \( I \) is completely determined by the three functions:

\[
E(u, v) = \langle X_u, X_u \rangle \\
F(u, v) = \langle X_u, X_v \rangle \\
G(u, v) = \langle X_v, X_v \rangle
\]
Proposition 1. Let $R \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization $X : U \rightarrow S$. Then

$$\int_{X^{-1}(R)} \|X_u \times X_v\| \, dudv = \text{Area}(R).$$
Proposition 1. Let \( R \subset S \) be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization \( X : U \rightarrow S \). Then

\[
\int_{X^{-1}(R)} \|X_u \times X_v\|dudv = \text{Area}(R).
\]

Note that

\[
\|X_u \times X_v\|^2 + \langle X_u, X_v \rangle^2 = \|X_u\|^2\|X_v\|^2,
\]

so the above integrand can be written as

\[
\|X_u \times X_v\| = \sqrt{EG - F^2}.
\]
The Gauss Map and curvature

At a point $p \in S$, define the normal vector to $p$ by the map $N : S \to S^2$ given by

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p).$$

This map is called the Gauss map.
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**Remark:** The differential $dN_p : T_pS \to T_pS^2 \simeq T_pS$ defines a self-adjoint linear map. Define

$$K = \det dN_p \text{ and } H = \frac{-1}{2}\text{trace } dN_p,$$

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Note: Since \( dN_p \) is self-adjoint, it can be diagonalized:
\[
dN_p = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.
\]
\( k_1 \) and \( k_2 \) are called the principal curvatures of \( S \) at \( p \) and the associated eigenvectors are called the principal directions.
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If we give $S^2$ the opposite orientation (i.e. choose the inward normal instead of the outward normal), then

$$dN_p = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $N$ is a conformal map.
The Catenoid

Figure 1: The catenoid is a minimal surface
Figure 2: The helicoid is a minimal surface as well
The helicoid and the catenoid are locally isometric
Helicoid with genus
Helicoid with genus
Catenoid Fence
Riemann’s minimal surface
Singly-periodic Scherk surface
Doubly-periodic Scherk surface
Fundamental domain for Scherk’s surface
Sherk’s surface with handles
Triply-periodic surface (Schwarz)
Why are these surfaces “minimal”? 

From this definition, it’s not at all clear what is “minimal” about these surfaces.

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Figure 3: The surface and a normal variation
Normal Variations

Let \( X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a regular parametrized surface. Let \( D \subset U \) be a bounded domain and let \( h : \bar{D} \rightarrow \mathbb{R} \) be differentiable. Then the normal variation of \( X(\bar{D}) \) determined by \( h \) is given by

\[
\varphi : \bar{D} \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3
\]

where

\[
\varphi(u, v, t) = X(u, v) + th(u,v)N(u,v).
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For small \( \epsilon \), \( X^t(u, v) = \varphi(u, v, t) \) is a regular parametrized surface with

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X^t_u = X_u + th_Nu + th_uN
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$$X^t_u = X_u + thN_u + th_u N$$
$$X^t_v = X_v + thN_v + th_v N.$$ 

Then it’s straightforward to see that

$$E^t G^t - (F^t)^2 = EG - F^2 - 2th(Eg - 2Ff + Ge) + R$$
$$= (EG - F^2)(1 - 4thH) + R$$

where $\lim_{t \rightarrow 0} \frac{R}{t} = 0$. 

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Equivalence of curvature and variational definitions of minimal

The area $A(t)$ of $X^t(\bar{D})$ is given by

$$A(t) = \int_{\bar{D}} \sqrt{E^tG^t - (F^t)^2} dudv$$

$$= \int_{\bar{D}} \sqrt{1 - 4thH + \bar{R}\sqrt{EG - F^2}} dudv,$$

where $\bar{R} = \frac{R}{EG - F^2}$.
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Hence, if $\epsilon$ is small, $A$ is differentiable and

$$A'(0) = - \int_{\bar{D}} 2hH \sqrt{EG - F^2} dudv.$$
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Therefore:

**Proposition 2.** Let $X : U \rightarrow \mathbb{R}^3$ be a regular parametrized surface and let $D \subset U$ be a bounded domain. Then $X$ is minimal (i.e. $H \equiv 0$) if and only if $A'(0) = 0$ for all such $D$ and all normal variations of $X(\bar{D})$. 
Isothermal coordinates

Definition 2. Given a regular surface $S \subset \mathbb{R}^3$, a system of local coordinates $X : U \rightarrow V \subset S$ is said to be isothermal if

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle \text{ and } \langle X_u, X_v \rangle = 0.$$  

i.e. $G = E$ and $F = 0$. 
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**Note:** A parametrization \( X : U \to V \subset S \) is isothermal if and only if it is conformal.
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Facts about isothermal parametrizations

Proposition 3. Let $X : U \rightarrow V \subset S$ be an isothermal local parametrization of a regular surface $S$. Then

$$\Delta X = X_{uu} + X_{vv} = 2\lambda^2 H,$$

where $\lambda^2(u,v) = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle$ and $H = HN$ is the mean curvature vector field.
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**Corollary 4.** Let $X(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$ be an isothermal parametrization of a regular surface $S$ in $\mathbb{R}^3$. Then $S$ is minimal (within the range of this parametrization) if and only if the three coordinate functions $x^1(u, v), x^2(u, v)$ and $x^3(u, v)$ are harmonic.
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**Theorem 6.** Isothermal coordinates exist on any minimal surface $S \subset \mathbb{R}^3$.

**Remark:** In fact, isothermal coordinates exist for any $C^2$ surface.
Connections with complex analysis

If $\sigma : S^2 \rightarrow \mathbb{C} \cup \{\infty\}$ is stereographic projection and $N : S \rightarrow S^2$ is the Gauss map, then

$$g := \sigma \circ N : S \rightarrow \mathbb{C} \cup \{\infty\}$$

is orientation-preserving and conformal whenever $dN \neq 0$. Therefore $g$ is a meromorphic function on $S$ (now thought of as a Riemann surface).
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In fact, we have:

**Theorem 7** (Osserman). *If \( S \) is a complete, immersed minimal surface of finite total curvature, then \( S \) can be conformally compactified to a Riemann surface \( \Sigma_k \) by closing finitely many punctures. Moreover, the Gauss map \( N : S \to S^2 \), which is meromorphic, extends to a meromorphic function on \( \Sigma_k \).*
Enneper’s Surface

Figure 4: $z \in \mathbb{C}$, $g(z) := z$
Chen-Gackstatter Surface
Chen-Gackstatter Surface with higher genus
Symmetric 4-Noid
Costa surface
Meeks’ minimal Möbius strip
Thanks!