

ON HOMOTOPY BRUNNIAN LINKS AND THE κ -INVARIANT

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ABSTRACT. We provide an alternative proof that Koschorke’s κ -invariant is injective on the set of link homotopy classes of n -component homotopy Brunnian links $BLM(n)$. The existing proof (by Koschorke [22]) is based on the Pontryagin–Thom theory of framed cobordisms, whereas ours is closer in spirit to techniques based on Habegger and Lin’s string links. We frame the result in the language of the rational homotopy Lie algebra $\pi_*(\Omega\text{Conf}(n)) \otimes \mathbb{Q}$ of the configuration space, which allows us to express Milnor’s μ -invariants as homotopy periods of $\text{Conf}(n)$.

1. INTRODUCTION

Consider the space of n -component link maps, i.e. smooth pointed maps

$$L : S^1 \sqcup \dots \sqcup S^1 \longrightarrow \mathbb{R}^3, \quad \text{such that } L_i(S^1) \cap L_j(S^1) = \emptyset, \quad i \neq j. \quad (1.1)$$

Two link maps L and L' are *link homotopic* if and only if there exists a smooth homotopy $H : (\bigsqcup_{i=1}^n S^1) \times I \mapsto \mathbb{R}^3$ connecting L and L' through link maps. Following [22], we denote the set of equivalence classes of n -component link maps by $LM(n)$. Link homotopy, originally introduced by Milnor [29], can be thought of as a crude equivalence relation on links, which in addition to standard isotopies allows for any given component to pass through itself. Link homotopies can be realized on diagrams via Reidemeister moves and finitely many crossing changes within each component. In particular, observe that link homotopy theory is completely trivial for knots. Equivalently, we may consider the space of free link maps, as there is a bijective correspondence between the pointed and basepoint free theory (see Appendix A).

In [29], Milnor classified 3-component links up to link homotopy by the set of invariants built from the pairwise linking numbers $\bar{\mu}(1; 2), \bar{\mu}(1; 3), \bar{\mu}(2; 3)$ and a triple linking number $\bar{\mu}(1, 2; 3)$, which is an integer modulo the indeterminacy given by $\gcd\{\bar{\mu}(1; 2), \bar{\mu}(1; 3), \bar{\mu}(2; 3)\}$. Further, Milnor defined *higher linking numbers* [30] as integer invariants modulo the indeterminacy given by lower order invariants. However, it is well known that these invariants are insufficient to classify $LM(n)$ for $n \geq 4$ (c.f. [23, 14]). A refinement of the $\bar{\mu}$ -invariants, due to Levine [23], classifies links up to four components.

1.1. Preliminaries. Let us briefly review Milnor’s definition of the $\bar{\mu}$ -invariants from [30]. The free group on generators $\{\tau_1, \dots, \tau_n\}$ will be denoted by $F(\tau_1, \dots, \tau_n)$ (or $F(n)$ without listing the generators). Given a group G , its k th lower central subgroup is defined inductively by $G_1 = G, G_2 = [G, G], \dots, G_k = [G, G_{k-1}]$.

Let L be a fixed n -component link in S^3 ; it has been shown in [2, 30] that, for any $q > 0$, the quotient $\mathcal{G}_q = \pi_1(S^3 - L)/(\pi_1(S^3 - L)_q)$ is an isotopy invariant of L and is generated by n meridians m_1, \dots, m_n , one for each component of L . Denote by w_1, \dots, w_n the words in \mathcal{G}_q

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representing untwisted longitudes of the components of L ; then \mathcal{G}_q has the presentation (c.f. [30, p. 289])

$$\langle m_1, \dots, m_n \mid [m_1, w_1], \dots, [m_n, w_n], F(m_1, \dots, m_n)_q \rangle.$$

The homomorphism $M : F(m_1, \dots, m_n) \rightarrow \mathbb{Z}\langle\langle X_1, \dots, X_n \rangle\rangle$ into the ring $\mathbb{Z}\langle\langle X_1, \dots, X_n \rangle\rangle$ of formal power series in the non-commuting variables $\{X_1, \dots, X_n\}$ – called the *Magnus expansion* – is defined on generators in the following way:

$$M(m_i) = 1 + X_i, \quad M(m_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \dots$$

Consider the expansion of the longitude $w_j \in \mathcal{G}_q$ treated as a word in $F(m_1, \dots, m_n)$:

$$M(w_j) = 1 + \sum_I \mu(I; j) X_I, \tag{1.2}$$

where the summation ranges over the multiindices $I = (i_1, \dots, i_m)$ where $1 \leq i_r \leq n$, and $X_I = X_{i_1} \cdots X_{i_m}$ for $m > 0$. Thus, the above expansion defines for each multiindex I the integer $\mu(I; j)$, which does not depend on q if $q \geq m$. Let $\Delta(I) = \Delta(i_1, \dots, i_m)$ denote the greatest common divisor of $\mu(J; j)$ where J is a proper subset of I (up to cyclic permutations). Then Milnor’s invariant $\bar{\mu}(I; j)$ of L is the residue of $\mu(I; j)$ modulo $\Delta(I)$. It is well known that Milnor’s higher linking numbers $\bar{\mu}(I; j)$ are invariants of L up to concordance and up to link homotopy if the indices in I are all distinct.

Habegger and Lin [11] provide an effective procedure for distinguishing classes in $LM(n)$. In place of links they consider the group of *string links* $\mathcal{H}(n)$ and prove a *Markov-type* theorem, giving a set of moves by which a string link can vary without changing the link homotopy class of its closure. There are two types of moves: (1) *ordinary conjugation* in $\mathcal{H}(n)$; and (2) a *partial conjugation*, which amounts to conjugation in the normal factor of any semidirect product decomposition of $\mathcal{H}(n)$ induced by forgetting one strand. However, it still remains an open problem whether a complete set of “numerical” link homotopy invariants for $LM(n)$ can be defined. For instance, the authors of [28, 15, 24] address this question using, among other things, the perspective of Vassiliev finite-type invariants. An alternative view, which is rarely cited in this context but which we aim to advocate here, appears in the work on higher dimensional link maps by Koschorke [21, 22], Haefliger [12], Massey and Rolfsen [26], and others (e.g. [31]).

Following [21, 22], recall that in the case of classical links (i.e. one-dimensional links in \mathbb{R}^3), the κ -invariant is defined as

$$\begin{aligned} \kappa : LM(n) &\rightarrow [\mathbb{T}^n, \text{Conf}(n)], \\ \kappa([L]) &= [F_L], \quad F_L = L(1) \times \cdots \times L(n), \end{aligned} \tag{1.3}$$

where $[\mathbb{T}^n, \text{Conf}(n)]$ is the set of pointed homotopy classes of maps from the n -torus \mathbb{T}^n to the configuration space $\text{Conf}(n)$ of n distinct points in \mathbb{R}^3 :

$$\text{Conf}(n) = \{(x_1, \dots, x_n) \in (\mathbb{R}^3)^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Again, we consider pointed maps in (1.3) purely for convenience (see Appendix A). Note that because any link homotopy between two links L_1 and L_2 induces a homotopy of the associated maps F_{L_1} and F_{L_2} , the map κ is well-defined. Koschorke [21, 22, 20] introduced the following central question which is directly related to the problems mentioned above.

Question 1.1 ([21, 22, 20]). *Is the κ -invariant injective and therefore a complete invariant of n -component classical links up to link homotopy?*

This question has been answered in the affirmative in the case of homotopy Brunnian links in [22, Theorem 6.1 and Corollary 6.2], and recently in the case of general 3-component links in [7, 8, 6].

1.2. Statement of the main result. Recall that an n -component link L is *homotopy Brunnian* [22] whenever all of its $(n - 1)$ -component sublinks are link-homotopically trivial. Denote the set of link homotopy classes of homotopy Brunnian n -links by $BLM(n) \subset LM(n)$. One of the clear advantages of working with $BLM(n)$ is that the indeterminacy of Milnor's higher linking numbers vanishes for such links, and in fact they constitute a complete set of invariants for $BLM(n)$ (this was first proved in [29]). In order to frame the main theorem in the language of the rational Lie algebra of the based loops on the configuration space $\Omega\text{Conf}(n)$ we first introduce the necessary background.

In the following, $\Omega\text{Conf}(n)$ denotes the based loop space of the configuration space $\text{Conf}(n)$. It is well known [4, 9] that $\mathcal{L}(\text{Conf}(n)) = \pi_*(\Omega\text{Conf}(n)) \otimes \mathbb{Q}$ is a graded Lie algebra with the bracket given by the Samelson product which is the adjoint of the usual Whitehead product [35]. More precisely, for any space X the Samelson product of $\alpha \in \pi_k(\Omega X)$ and $\beta \in \pi_j(\Omega X)$, is given by

$$[\alpha, \beta] = (-1)^{k-1} \partial_* [\partial_*^{-1} \alpha, \partial_*^{-1} \beta]_W,$$

where $\partial_* : \pi_{p+1}(X) \rightarrow \pi_p(\Omega X)$ is the adjoint homomorphism and $[\cdot, \cdot]_W$ is the Whitehead product. The generators of $\mathcal{L}(\text{Conf}(n))$ are all in degree 1 and are represented by maps $B_{j,i} : S^1 \rightarrow \Omega\text{Conf}(n)$, $1 \leq i < j \leq n$, defined as adjoints of pointed versions of the spherical cycles

$$A_{j,i} : S^2 = \Sigma S^1 \longrightarrow \text{Conf}(n), \quad A_{j,i}(\xi) = (\dots, \underbrace{q_j}_{i\text{'th}}, \dots, \underbrace{q_j + \xi}_{j\text{'th}}, \dots), \quad \xi \in S^2, \quad (1.4)$$

where $q = (q_1, \dots, q_n)$ is fixed in $\text{Conf}(n)$. We also have the following vector space isomorphism [9, p. 22]:

$$\mathcal{L}(\text{Conf}(n)) \stackrel{\text{vect.}}{\cong} \bigoplus_{j=1}^{n-1} \pi_*(\underbrace{\Omega(S^2 \vee \dots \vee S^2)}_{j \text{ times}}) \otimes \mathbb{Q} = \bigoplus_{j=1}^{n-1} \mathcal{L}(B_{j+1,1}, B_{j+1,2}, \dots, B_{j+1,j}), \quad (1.5)$$

where the j th factor $\pi_*(\Omega(S^2 \vee \dots \vee S^2)) \otimes \mathbb{Q}$ is the free Lie algebra generated by $\{B_{j+1,1}, B_{j+1,2}, \dots, B_{j+1,j}\}$. As a Lie algebra, $\mathcal{L}(\text{Conf}(n))$ is the quotient of the direct sum of the free Lie algebras $\mathcal{L}_j := \mathcal{L}(B_{j+1,1}, B_{j+1,2}, \dots, B_{j+1,j})$ by the *4T-relations* [9, 17, 4]

$$B_{i,j} = -B_{j,i} \quad [B_{\sigma(2),\sigma(1)}, B_{\sigma(4),\sigma(3)}] = 0, \quad (\text{for } n \geq 4), \quad (1.6)$$

$$[B_{\sigma(2),\sigma(1)}, B_{\sigma(3),\sigma(1)} + B_{\sigma(3),\sigma(2)}] = 0, \quad (1.7)$$

where σ is any permutation on $\{1, 2, \dots, n\}$. We can now state the main theorem of this paper:

Main Theorem. *The restriction of κ to $BLM(n)$ is injective. Moreover,*

- (i) *the image $\kappa(BLM(n))$ of $BLM(n)$ is contained in a copy of $\pi_n(\text{Conf}(n)) \cong \pi_{n-1}(\Omega\text{Conf}(n))$ inside $[\mathbb{T}^n, \text{Conf}(n)]$ and it is a free, rank $(n - 2)!$ module generated by the iterated Samelson products*

$$B(n, \sigma) = [B_{n,1}, B_{n,\sigma(2)}, \dots, B_{n,\sigma(n-1)}], \quad \sigma \in \Sigma(2, \dots, n - 1);$$

- (ii) *for any representative link $L \in BLM(n)$ we have the following expansion in the above basis:*

$$\kappa(L) = \sum_{\sigma \in \Sigma(2, \dots, n-1)} \mu(1, \sigma(2), \dots, \sigma(n-1); n) B(n, \sigma).$$

Recall [34, 13] that the *homotopy periods* of a simply connected manifold M (with finite Betti numbers) are integrals in the differential forms on M which detect all nontrivial elements of $\pi_*(M) \otimes \mathbb{Q}$. It is well known that the homotopy periods problem is completely solvable; see [34, 13] and the recent work [33]. As a direct consequence of (ii) we obtain

Corollary 1.2. *Given any $L \in BLM(n)$, the associated μ -invariants[†] $\{\mu(1, \sigma(2), \dots, \sigma(n-1); n)(L)\}_\sigma$, $\sigma \in \Sigma(2, \dots, n-1)$ are fully determined by the homotopy periods of the basis elements $B(n, \sigma)$.*

The inspiration for the proof of the Main Theorem comes from the techniques introduced by the first author in [3] and from the algebraic proof of [7, 8]. This seems like a promising approach for answering Question 1.1, not least because a generalization of the $n = 3$ case suffices to completely answer the question for 3-component links (see in particular [7, Section 5]). Moreover, Corollary 1.2 implies that the μ -invariants of homotopy Brunnian links can be computed by Chen's iterated integrals [13], which in this light appear as generalized Gauss integrals and hence as a possible source of invariants for fluid flows [1, p. 176] (see also [18, 19]).

This paper is structured as follows. In Section 2, we gather basic facts about string links and use them to provide an algebraic characterization of $BLM(n)$. Then, in Section 3 we review algebraic properties of the torus homotopy groups $[\mathbb{T}^{n-1}, \Omega\text{Conf}(n)]$ and, in anticipation of the proof of the Main Theorem, recognize an analogue of $BLM(n)$ in this context. Finally, we prove the theorem in Section 4. By the usual abuse of notation we often do not make a distinction between equivalence classes and their representatives.

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2. STRING LINKS

Following Habegger and Lin [11], let $\mathcal{H}(n)$ be the group of link homotopy classes of ordered, oriented, parametrized string links with n components. There is a split short exact sequence of groups

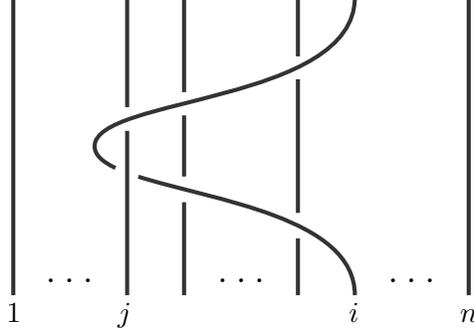
$$1 \longrightarrow \mathcal{C}(n; i) \longrightarrow \mathcal{H}(n) \xrightarrow{\delta_i} \mathcal{H}(n; i) \longrightarrow 1 \quad (2.1)$$

where $\mathcal{H}(n; i)$ is the copy of $\mathcal{H}(n-1)$ given by the map δ_i which deletes the i th strand. The normal subgroup $\mathcal{C}(n; i)$ is isomorphic to $RF(n-1)$, the reduced free group on the $n-1$ generators shown in Figure 1.

The exact sequence (2.1) is split by the map $s_i : \mathcal{H}(n; i) \longrightarrow \mathcal{H}(n)$ which just adds one trivial strand to any element of $\mathcal{H}(n; i)$ as the i th strand. As a result we have the semidirect product decomposition (for each i)

$$\mathcal{H}(n) = \mathcal{H}(n; i) \rtimes \mathcal{C}(n; i). \quad (2.2)$$

[†]Following the convention in [30], we use μ rather than $\bar{\mu}$ here because the indeterminacy vanishes.


 FIGURE 1. A generator $\tau_{i,j}$ of $\mathcal{C}(n; i) \cong RF(n-1)$.

For $\rho \in \mathcal{H}(n)$, let

$$\rho = (\theta, h)_i = \theta_i h_i, \quad \theta_i \in \mathcal{H}(n; i) \quad \text{and} \quad h_i \in \mathcal{C}(n; i), \quad (2.3)$$

be the factorization of ρ with respect to the above decomposition. A *partial conjugation* of ρ by $\lambda \in \mathcal{H}(n)$ is obtained by $\rho = (\theta, h)_i \mapsto (\theta, \lambda h \lambda^{-1})_i$. The Markov-type theorem for homotopy string links is then

Theorem 2.1 (Habegger–Lin Classification Theorem [11]). *Denote the Markov closure operation by*

$$\widehat{\cdot} : \mathcal{H}(n) \longrightarrow LM(n). \quad (2.4)$$

Then

- (a) $\widehat{\cdot}$ is surjective.
- (b) $\widehat{\rho}_1 = \widehat{\rho}_2$ if and only if ρ_1 and ρ_2 in $\mathcal{H}(n)$ are related by a sequence of conjugations and partial conjugations (in fact, partial conjugations are sufficient [16]).

Some known facts about $\mathcal{H}(n)$ are:

- (1) $\mathcal{H}(n)$ is torsion-free and nilpotent of class $n-1$,
- (2) $\mathcal{H}(n)_k = \mathcal{H}(n; i)_k \times \mathcal{C}(n; i)_k$,
- (3) $\mathcal{H}(n)_{k-1} / \mathcal{H}(n)_k$ is a free abelian group of rank $(k-2)! \binom{n}{k}$.

In the following we will focus on the copy of $\mathcal{C}(n; n) \cong RF(n-1)$ in $\mathcal{H}(n)$ generated by $\langle \tau_1, \dots, \tau_{n-1} \rangle$ where $\tau_k := \tau_{n,k}$. Recall that the reduced free group $RF(n-1)$ is the quotient of $F(n-1)$ given by adding the relations $[\tau_i, g \tau_i g^{-1}] = 1$ for all i and all $g \in F(n-1)$. We list known and useful facts (see [3]) about $RF(n-1)$ below.

- (4) $RF(n-1)$ admits the presentation

$$\left\langle \tau_1, \dots, \tau_{n-1} \mid \text{for all } 1 \leq i_1 < \dots < i_k \leq n, 1 < k \leq n : \right. \\ \left. [\tau_{i_1}, \dots, \tau_{i_k}] = 1, \text{ whenever } \tau_{i_s} = \tau_{i_r} \text{ for some } s < r \right\rangle, \quad (2.5)$$

where $[\tau_{i_1}, \dots, \tau_{i_k}]$ denotes the simple k -fold commutator $[\dots [[\tau_{i_1}, \tau_{i_2}], \tau_{i_3}], \dots, \tau_{i_k}]$ (c.f. [25, p. 295]).

Denote by $I = (i_1, \dots, i_k)$ an *ordered* multiindex, where $1 \leq i_1 < \dots < i_k \leq n-1$, $1 \leq k \leq n-1$, and let

$$\tau(I, \sigma) := [\tau_{i_1}, \tau_{i_{\sigma(2)}}, \dots, \tau_{i_{\sigma(k)}}], \quad (2.6)$$

where σ is a permutation of $\{2, \dots, k\}$. Then

(5) $RF(n-1)_n = \{1\}$ and

$$RF(n-1)_k / RF(n-1)_{k+1} \cong \bigoplus_{(k-1)! \binom{n-1}{k}} \mathbb{Z}.$$

(6) Each $RF(n-1)_k / RF(n-1)_{k+1}$ is generated by $\{\tau(I, \sigma)\}$, with $|I| = k$, and elements $z \in RF(n-1)$ have the normal form

$$z = \lambda_1 \lambda_2 \cdots \lambda_{n-1}, \quad \text{where } \lambda_k = \prod_{I, |I|=k} \prod_{\sigma \in \Sigma(2, \dots, k)} \tau(I, \sigma)^{e(I, \sigma)} \quad \text{for some } e(I, \sigma) \in \mathbb{Z}. \quad (2.7)$$

Consider the homomorphism

$$\delta : \mathcal{H}(n) \longrightarrow \prod_{i=1}^n \mathcal{H}(n; i), \quad \delta = \prod_{i=1}^n \delta_i, \quad (2.8)$$

where δ_i is as defined in (2.1). Clearly,

$$\ker \delta = \mathcal{C}(n; 1) \cap \mathcal{C}(n; 2) \cap \dots \cap \mathcal{C}(n; n). \quad (2.9)$$

Observe that elements of $\ker \delta$ have a natural geometric meaning: they are precisely the string links which become trivial after removing *any* of their components, which we naturally call *Brunnian string links* and denote by $B\mathcal{H}(n)$; i.e., $B\mathcal{H}(n) := \ker \delta$. In the ensuing lemma we choose to treat $B\mathcal{H}(n)$ as a subgroup of $\mathcal{C}(n; n) \cong RF(\tau_1, \dots, \tau_{n-1})$.

Lemma 2.2. *$B\mathcal{H}(n)$ is a free abelian group of rank $(n-2)!$ generated by*

$$\tau(n, \sigma) := [\tau_1, \tau_{\sigma(2)}, \dots, \tau_{\sigma(n-1)}] \quad \text{for } \sigma \in \Sigma(2, \dots, n-1). \quad (2.10)$$

Moreover,

- (i) $B\mathcal{H}(n) = \mathcal{C}(n; i)_{n-1} \cong RF(n-1)_{n-1}$, $B\mathcal{H}(n) = \mathcal{H}(n)_{n-1}$
- (ii) $B\mathcal{H}(n) \subset Z(\mathcal{H}(n))$.

Proof. The fact that $\mathcal{H}(n)_{n-1} \subset Z(\mathcal{H}(n))$ follows immediately from nilpotency (i.e. length n commutators are all trivial in $\mathcal{H}(n)$). Thus (i) implies (ii).

Clearly $B\mathcal{H}(n) \subset \ker \delta_n = \mathcal{C}(n; n)$, so any $z \in B\mathcal{H}(n)$ can be written in the normal form (2.7). We claim that $z = \lambda_{n-1}$. Indeed, for each $k < n-1$ and any $I = (i_1, \dots, i_k)$, consider $\tau(I, \sigma)$ given in (2.6). Pick $j \in \{1, \dots, n-1\}$ such that $j \neq i_r$ for all $i_r \in I$, which is possible since $k < n-1$. The map $\delta_j : \mathcal{C}(n; n) \rightarrow \mathcal{C}(n; n, j)$ which deletes the j th strand is given on generators as

$$\delta_j(\tau_i) = \begin{cases} 1 & \text{for } j = i, \\ \tau_i & \text{otherwise,} \end{cases} \quad (2.11)$$

so we have $\delta_j(\tau(I, \sigma)) = \tau(I, \sigma)$. Therefore, $\delta_j(\lambda_k) \neq 1$ for all $j = 1, \dots, n-1$, contradicting the fact that $z \in \ker \delta_j$ for all j . Hence, the normal form (2.7) implies $z = \lambda_{n-1}$ and therefore (2.10) follows as well as the first part of (i).

The second identity in (i) is immediate: (2) implies that $\mathcal{H}(n)_{n-1} = \mathcal{H}(n; i)_{n-1} \times \mathcal{C}(n; i)_{n-1}$, but this is just $\mathcal{C}(n; i)_{n-1}$ since $\mathcal{H}(n; i)_{n-1}$ is trivial by (1). \square

Remark 2.3. The proof of the fact that $\{\tau(n, \sigma)\}$ generates $B\mathcal{H}(n)$ is fully analogous to that of claim (a) in Appendix B.

The relation between Brunnian string links and Brunnian links is revealed in the following result, which is a consequence of Lemma 2.2 and Theorem 2.1.

Proposition 2.4. *The restriction of the Markov closure operation defined in (2.4) to $B\mathcal{H}(n)$ is injective, and the image $\widehat{B\mathcal{H}(n)}$ equals $BLM(n)$.*

Proof. In order to see that $\widehat{\cdot}$ is injective on $B\mathcal{H}(n)$, let $\rho \in B\mathcal{H}(n)$. Since $B\mathcal{H}(n) = \mathcal{C}(n; i)_{n-1}$, the factorization

$$\rho = (\theta, h)_i = \theta_i h_i$$

is only valid if $\theta_i = 1$. Hence, all partial conjugations of ρ are just ordinary conjugations and, since $B\mathcal{H}(n) \subset Z(\mathcal{H}(n))$, they act trivially on ρ . Thus Markov closure is injective on $B\mathcal{H}(n)$.

In order to see $\widehat{B\mathcal{H}(n)} = BLM(n)$, observe that the inverse image of $[1] \in LM(m)$ under $\widehat{\cdot} : \mathcal{H}(m) \rightarrow LM(m)$ contains only $1 \in \mathcal{H}(m)$. Indeed, Theorem 2.1(b) tells us that any element of the inverse image of $[1]$ under $\widehat{\cdot}$ has to be related to 1 by conjugations or partial conjugations. Obviously, both of these operations act trivially on 1, which proves the claim.

Let us fix a representative link $L \in BLM(n)$ and by Theorem 2.1(a) let $\rho \in \mathcal{H}(n)$ be such that $\widehat{\rho} = L$. Since any $(n-1)$ -component sublink $\widehat{\delta_i(\rho)}$ of L is trivial, the fact proven above implies that $\delta_i(\rho) = 1$ for any $1 \leq i \leq n$, and therefore $\rho \in \ker \delta = B\mathcal{H}(n)$. \square

3. TORUS HOMOTOPY GROUPS

In this and the following sections all sets of homotopy classes $[X, Y]$ are pointed and $*$ denotes a basepoint. Consider the group

$$T(n) := [\Sigma \mathbb{T}^{n-1}, \text{Conf}(n)] = [\mathbb{T}^{n-1}, \Omega \text{Conf}(n)] \quad (3.1)$$

of homotopy classes of pointed maps from the $(n-1)$ -torus $\mathbb{T}^{n-1} = \underbrace{S^1 \times \dots \times S^1}_{n-1 \text{ times}}$ to the based

loop space $\Omega \text{Conf}(n)$ of the configuration space $\text{Conf}(n)$. The product in $T(n)$ comes from the loop multiplication or equivalently the coproduct of suspensions. In the notation of Fox, who introduced torus homotopy groups [10], $T(n)$ is denoted by $\tau_n(\text{Conf}(n))$ and equivalently defined as $\pi_1(\text{Maps}(\mathbb{T}^{n-1}, \text{Conf}(n)), *)$, where the basepoint $*$ in the case of $T(n)$ is defined to be the constant map. In the following, we will freely alternate between both ways of representing $T(n)$ given in (3.1). Letting P_k be the k -skeleton of \mathbb{T}^{n-1} , we have the filtration

$$\{s\} = P_0 \subset P_1 \subset \dots \subset P_{n-1} = \mathbb{T}^{n-1}, \quad * = s = (s_1, \dots, s_{n-1}).$$

For each ordered multiindex I , observe that $P_k = \bigcup_{|I|=k} S_I$, where

$$S_I = \{t = (t_1, \dots, t_{n-1}) \in \mathbb{T}^{n-1} \mid t_i = s_i \text{ for } i \notin I\}. \quad (3.2)$$

We have a decreasing filtration of groups

$$T(n) \supset T(n)_1 \supset \dots \supset T(n)_{n-1} = \{1\}, \quad \text{where } T(n)_k \cong [(\mathbb{T}^{n-1}, P_k); (\Omega \text{Conf}(n), *)].$$

Some known facts about $\{T(n)_k\}$ are summarized below (c.f. [35, p. 462]):

- (7) $\{T(n)_k\}$ is a central chain of $T(n)$.
- (8) $T(n)$ is nilpotent of class $n-1$.

(9)

$$T(n)_{k-1} / T(n)_k \cong \bigoplus_{I, |I|=k-1} [S^I, \Omega\text{Conf}(n)] = \bigoplus_{\binom{n-1}{k-1}} \pi_k(\Omega\text{Conf}(n)), \quad (3.3)$$

where $S^I := S_I / (S_I \cap P_{|I|-1})$.

Each factor $\pi_k(\Omega\text{Conf}(n))$ in the direct sum of (3.3) turns out to be a subgroup of $T(n)$ via a monomorphism indexed by $I = (i_1, \dots, i_k)$, $1 \leq i_1 < \dots < i_k \leq n-1$ defined as follows (c.f. [10, 35]):

$$\begin{aligned} j_I^\# : [S^I, \Omega\text{Conf}(n)] &\longrightarrow T(n), \\ j_I^\#([f]) &= [f \circ j_I], \quad j_I : \mathbb{T}^{n-1} \longrightarrow S^I, \end{aligned} \quad (3.4)$$

where j_I is the quotient projection. We introduce the following notation

$$\pi_I(\Omega\text{Conf}(n)) := [S^I, \Omega\text{Conf}(n)] \cong \pi_{|I|}(\Omega\text{Conf}(n)).$$

Another way to see (3.3) is via the homotopy equivalence

$$\Sigma\mathbb{T}^{n-1} \cong \bigvee_{k=1}^{n-1} \bigvee_{I, |I|=k} \Sigma S^k. \quad (3.5)$$

Then, each monomorphism $j_I^\#$ of (3.4) can be regarded as induced from the restriction in the above bouquet to the I th factor $\Sigma S^{|I|}$ of $\Sigma\mathbb{T}^{n-1}$. We have the following lemma which is a special case of a lemma due to Fox [10]:

Lemma 3.1 ([10, 35]). *Given multiindices I, J :*

- (i) *Suppose $I \cap J = \emptyset$ and $\epsilon = (-1)^w$, where w is the number of instances of $i > j$ with $i \in I$ and $j \in J$. Then for any $\alpha \in \pi_I(\Omega\text{Conf}(n)), \beta \in \pi_J(\Omega\text{Conf}(n))$, we have*

$$j_{I \cup J}^\#([\alpha, \beta]) = [j_I^\#(\alpha), j_J^\#(\beta)]^\epsilon,$$

where $[\alpha, \beta]$ is the Samelson product of α and β , and $[,]$ on the right hand side denotes the commutator in $T(n)$.

- (ii) *If $I \cap J \neq \emptyset$, then*

$$[j_I^\#(\alpha), j_J^\#(\beta)] = 1.$$

- (iii) *For every ordered multiindex I , $j_I^\#$ is a monomorphism.*

Now, we aim to understand the free part of $T(n)$, which we will be denoted by $TF(n)$. Consider the following elements of $T(n)$ obtained from the generators $\{B_{j,i}\}$ of $\mathcal{L}(\text{Conf}(n))$, given as adjoints of $\{A_{j,i}\}$ in (1.4):

$$t_{k,i}(\ell) := j_{(\ell)}^\#(B_{k,i}) \in \pi_{(\ell)}(\Omega\text{Conf}(n)), \quad 1 \leq i < k \leq n, \quad 1 \leq \ell \leq n-1, \quad (3.6)$$

where (ℓ) is the length 1 multiindex. In view of the fact that $\{B_{k,i}\}$ are free generators of $\pi_1(\Omega\text{Conf}(n))$, $\{t_{k,i}(\ell)\}$ are infinite cyclic and generate a subgroup of $T(n)$ which we call the *free part* of $T(n)$ and denote by $TF(n)$.

Lemma 3.2. *The lower central series $\{TF(n)_k\}$ of $TF(n)$ has the following properties:*

- (10) *$\{TF(n)_k\}$ is nilpotent of class $n-1$.*
(11) *$TF(n)_k \subset T(n)_k$ for every k .*

(12)

$$TF(n)_{k-1} / TF(n)_k \cong \bigoplus_{I, |I|=k-1} \pi_I(\Omega\text{Conf}(n))^{\text{free}}. \quad (3.7)$$

Proof. By Theorem 5.4 of [25], the k th stage $TF(n)_k$ of the lower central series of $TF(n)$ is generated by the following simple r -fold commutators for $r \geq k$:

$$[t_{j_1, i_1}(\ell_1), t_{j_2, i_2}(\ell_2), \dots, t_{j_r, i_r}(\ell_r)], \quad 1 \leq i_s < j_s \leq n, \quad 1 \leq \ell_s \leq n-1, \quad 1 \leq s \leq r, \quad (3.8)$$

with no additional assumptions on the order or distinctiveness of (j_1, \dots, j_r) , (i_1, \dots, i_r) , and (ℓ_1, \dots, ℓ_r) . For (10), observe that if $k = n$, any multiindex (ℓ_1, \dots, ℓ_n) has to have repeating entries, since $1 \leq \ell_s \leq n-1$ for $1 \leq s \leq n$. Then Lemma 3.1(ii) tells us that any n -fold commutator as in (3.8) is trivial, proving (10). Moreover, from Lemma 3.1(i) and (3.6),

$$j_{(\ell_1, \dots, \ell_r)}^\#(\pm[B_{j_1, i_1}, B_{j_2, i_2}, \dots, B_{j_r, i_r}]) = [t_{j_1, i_1}(\ell_1), t_{j_2, i_2}(\ell_2), \dots, t_{j_r, i_r}(\ell_r)],$$

implying that

$$[t_{j_1, i_1}(\ell_1), t_{j_2, i_2}(\ell_2), \dots, t_{j_r, i_r}(\ell_r)] \in \pi_{(\ell_1, \dots, \ell_r)}(\Omega\text{Conf}(n))^{\text{free}} \subset \pi_{(\ell_1, \dots, \ell_r)}(\Omega\text{Conf}(n)).$$

Thus, as a consequence of (9), we obtain (11). By the isomorphism (1.5), the converse is true as well; i.e., any element of $\pi_{(\ell_1, \dots, \ell_r)}(\Omega\text{Conf}(n))^{\text{free}}$ is an integer combination of Samelson products $\{[B_{j_1, i_1}, B_{j_2, i_2}, \dots, B_{j_r, i_r}]\}$. Identity (12) now follows, because any coset of $TF(n)_{k-1} / TF(n)_k$ is represented by a product of $(k-1)$ -fold commutators in (3.8). \square

In the remaining part of this section we will define and characterize the *Brunnian part* of $TF(n)$, which we denote by $BTF(n)$. Consider the following projection map, which is defined by analogy to the string link story as

$$\Omega\psi : \Omega\text{Conf}(n) \xrightarrow{\prod_{i=1}^n \Omega\psi_i} \prod_n \Omega\text{Conf}(n-1),$$

obtained by looping $\psi = \psi_1 \times \dots \times \psi_n$, where $\psi_i : \text{Conf}(n) \rightarrow \text{Conf}(n-1)$ is the projection

$$\psi_i : (x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, \hat{x}_i, \dots, x_n)$$

which deletes the i th coordinate. As in the case of string links, we say that $t \in TF(n)$ is *Brunnian* whenever t belongs to the kernel of the homomorphism

$$\Psi : TF(n) \longrightarrow \prod_n [\mathbb{T}^{n-1}, \Omega\text{Conf}(n-1)],$$

given as a product $\Psi = \prod_{i=1}^{n-1} \Omega\psi_i^\#$, where $\Omega\psi_i^\# : [\mathbb{T}^{n-1}, \Omega\text{Conf}(n)] \longrightarrow [\mathbb{T}^{n-1}, \Omega\text{Conf}(n-1)]$ is induced by ψ_i for each i . The following result should seem like *d  j   vu* of Lemma 2.2 from the previous section (see Appendix B for the proof).

Lemma 3.3. *The group $BTF(n) = \ker \Psi$ is a free abelian group of rank $(n-2)!$ generated by iterated commutators in $TF(n)$ of the form*

$$t(n, \sigma) = [t_{n,1}(1), t_{n,\sigma(2)}(\sigma(2)), \dots, t_{n,\sigma(n-1)}(\sigma(n-1))], \quad (3.9)$$

where $\sigma \in \Sigma(2, \dots, n-1)$ is any permutation of $\{2, \dots, n-1\}$. In particular, it follows that

$$BTF(n) \subset TF(n)_{n-1} = \pi_N(\Omega\text{Conf}(n))^{\text{free}}, \quad (3.10)$$

where $N = (1, 2, \dots, n-1)$ is the top ordered index in the notation of (3.4).

4. PROOF OF MAIN THEOREM

For convenience let us restate our main result:

Main Theorem. *The restriction of κ to $BLM(n)$ is injective. Moreover,*

- (i) *the image $\kappa(BLM(n))$ of $BLM(n)$ is contained in a copy of $\pi_n(\text{Conf}(n))$ inside $[\mathbb{T}^n, \text{Conf}(n)]$ and it is a free, rank $(n-2)!$ module generated by the iterated Samelson products*

$$B(n, \sigma) = [B_{n,1}, B_{n,\sigma(2)}, \dots, B_{n,\sigma(n-1)}], \quad \sigma \in \Sigma(2, \dots, n-1); \quad (4.1)$$

- (ii) *for any representative link $L \in BLM(n)$ we have the following expansion in the above basis:*

$$\kappa(L) = \sum_{\sigma \in \Sigma(2, \dots, n-1)} \mu(1, \sigma(2), \dots, \sigma(n-1); n) B(n, \sigma).$$

Let $s = (0, \dots, 0)$ be the basepoint of \mathbb{T}^n and let each factor be parametrized by the unit interval. Distinguish the following subsets of \mathbb{T}^n :

$$A_t := \mathbb{T}^{n-1} \times \{t\}, \quad S_n := \{(0, \dots, 0)\} \times S^1, \quad (4.2)$$

and define the maps

$$\begin{aligned} p^\# : [\Sigma\mathbb{T}^{n-1}, \text{Conf}(n)] &\longrightarrow [\mathbb{T}^n, \text{Conf}(n)], & \text{where} \\ p : \mathbb{T}^n &\longrightarrow \mathbb{T}^n / (A_0 \vee S_n) \cong \Sigma\mathbb{T}^{n-1}. \end{aligned} \quad (4.3)$$

Lemma 4.1. *Let $N = (1, 2, \dots, n-1)$ be the top ordered multiindex. Consider the composition*

$$\pi_N(\Omega\text{Conf}(n)) \xrightarrow{j_N^\#} T(n) \xrightarrow{p^\#} [\mathbb{T}^n, \text{Conf}(n)],$$

where $j_N^\#$ is defined in (3.4). Then $p^\# \circ j_N^\#$ is injective.

This result follows from Satz 12, Satz 20 in [32] (see [22, p. 305]), but for completeness we provide an independent argument in Appendix C. Next, we turn to the proof of the Main Theorem.

We will work with $\mathcal{C}(n; n)$, which is a copy of $RF(n-1)$ inside $\mathcal{H}(n)$, and begin by constructing a homomorphism

$$\phi : \mathcal{C}(n; n) \longrightarrow TF(n) = [\Sigma\mathbb{T}^{n-1}, \text{Conf}(n)]^{\text{free}}$$

via the canonical homomorphism $F(\tau_1, \dots, \tau_{n-1}) \longrightarrow TF(n)$ defined on generators by

$$\phi : \tau_i \mapsto t_{n,i}(i).$$

Observe, by Lemma 3.1(ii), that any commutator in $\{t_{n,i}(i)\}$ with repeats is trivial. Therefore, as a direct consequence of the presentation in (4), we can pass to the quotient and obtain $\phi : \mathcal{C}(n; n) \longrightarrow TF(n)$, as required. We will need the following relation between the κ -invariant of (1.3) and ϕ .

Fact: For any $z \in \mathcal{C}(n; n)$ we have

$$\kappa(\widehat{z}) = p^\#(\phi(z)); \quad (4.4)$$

i.e., the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{C}(n; n) & \xrightarrow{\phi} & TF(n) \\
 \downarrow \widehat{\cdot} & & \downarrow p^\# \\
 LM(n) & \xrightarrow{\kappa} & [\mathbb{T}^n, \text{Conf}(n)].
 \end{array} \tag{4.5}$$

With this fact in hand consider the composition $\phi \circ \iota$ where $\iota : B\mathcal{H}(n) \rightarrow \mathcal{C}(n; n)$ is the inclusion monomorphism (see Lemma 2.2). The normal form of any $z \in B\mathcal{H}(n)$ is a product of terms $\tau(n, \sigma)$ as defined in (2.10); thus, by the definition of ϕ and (3.9) we immediately obtain

$$\phi(\tau(n, \sigma)) = t(n, \sigma).$$

Hence, Lemma 3.3 implies that $\phi \circ \iota$ is a monomorphism with image equal to $BTF(n)$. Further, the Fact stated above yields the commutative diagram

$$\begin{array}{ccc}
 B\mathcal{H}(n) & \xrightarrow{\phi \circ \iota} & BTF(n) \subset \pi_N(\Omega\text{Conf}(n)) \\
 \downarrow \widehat{\cdot} & & \downarrow j_N^\# \\
 BLM(n) & \xrightarrow{\kappa} & [T^n, \text{Conf}(n)].
 \end{array} \tag{4.6}$$

By Lemma 4.1 the composition $p^\# \circ j_N^\# \circ \phi \circ \iota$ is injective. Further, by Proposition 2.4 the Markov closure map $\widehat{\cdot}$ is a bijection on $B\mathcal{H}(n)$. Therefore, the injectivity of κ follows, proving (i) of the theorem modulo the above Fact. We also have the set identity

$$\kappa(BLM(n)) = p^\# \circ j_N^\#(BTF(n)),$$

which implies that $\kappa(BLM(n))$ has the structure of a Lie \mathbb{Z} -module with basis given by the iterated Samelson products $B(n, \sigma)$ from (4.1).

Proof of the Fact. As a first step, we show commutativity of (4.5) on the generators $\{\tau_i\}$ of $\mathcal{C}(n; n)$. Given a multiindex I , we adapt the notation S^I and S_I from (3.2) and **(9)**, respectively, to \mathbb{T}^n . Observe that, because all strands of $\widehat{\tau}_i$ except the n th and the i th can be collapsed to the basepoint (see Figure 1), $\kappa(\widehat{\tau}_i)$ factors, up to homotopy, through

$$\mathbb{T}^n \xrightarrow{p_{(i,n)}} S_{(i,n)} = S_i \times S_n \xrightarrow{\kappa(\widehat{\tau}_i)|_{S_i \times S_n}} \text{Conf}(n).$$

Since the i th and n th strands of τ_i link once, the restriction $\kappa(\widehat{\tau}_i)|_{S_i \times S_n}$ has degree 1 after composing with the projection $\Pi_{i,n} : \text{Conf}(n) \rightarrow \text{Conf}(2) \cong S^2$ onto the i th and n th coordinates. Further, $\kappa(\widehat{\tau}_i)|_{S_i \times S_n}$ factors through

$$S_i \times S_n \xrightarrow{p_i} \Sigma S_i \xrightarrow{A_{n,i}} \text{Conf}(n),$$

where the $A_{n,i}$ are defined in (1.4) (note that p_i induces a bijection $p_i^\# : [\Sigma S_i, \text{Conf}(n)] \rightarrow [S_i \times S_n, \text{Conf}(n)]$). Using the definitions in (3.4), (4.3) and (3.6) we have

$$\kappa(\widehat{\tau}_i) = p_{(i,n)}^\#(p_i^\#(A_{n,i})) = p^\#(j_{(i)}^\#(B_{n,i})) = p^\#(t_{n,i}(i)), \tag{4.7}$$

where the second equality is obtained by passing to adjoints. This shows that Diagram 4.5 commutes on the generators.



FIGURE 2. A combed representative of $[\tau_{3,2}, \tau_{3,1}] \in \mathcal{C}(3;3) \subset \mathcal{H}(3)$. A string link in $\mathcal{C}(n;n)$ is *combed* if all but one of the strands are trivial and the non-trivial strand rises monotonically from bottom to top.

Now, let $\tau \in \mathcal{C}(n;n)$ be a word of length k in $\{\tau_i\}$; specifically

$$\tau = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_k}. \quad (4.8)$$

Working with a combed representative of τ (as in Figure 2), we let

$$0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1$$

be such that the restriction of the t parameter to $[t_{j-1}, t_j]$ parametrizes τ_{i_j} in (4.8). Recall from (4.2) the subsets $A_j := A_{t_j}$ and S_n , which are the $(n-1)$ -torus with fixed last coordinate t_j and the n th coordinate circle, respectively. We have the following cofibration diagram together with the induced exact sequence of sets and groups:

$$\begin{array}{ccccc} (A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k) \cup S_n & \xrightarrow{\subset} & \mathbb{T}^n & \xrightarrow{h} & \bigvee_k \Sigma \mathbb{T}^{n-1} \\ \left[\bigsqcup_{i=1}^k A_i, \text{Conf}(n) \right] \times \pi_1 \text{Conf}(n) & \xleftarrow{\subset^\#} & [\mathbb{T}^n, \text{Conf}(n)] & \xleftarrow{h^\#} & \left[\bigvee_k \Sigma \mathbb{T}^{n-1}, \text{Conf}(n) \right] \end{array}$$

Since $\kappa(\widehat{\tau})$ restricted to any A_j is null and $\text{Conf}(n)$ is simply-connected, there exists some $z \in [\bigvee_k \Sigma \mathbb{T}^{n-1}, \text{Conf}(n)]$ such that $h^\#(z) = \kappa(\widehat{\tau})$. Let $r_j : \bigvee_k \Sigma \mathbb{T}^{n-1} \rightarrow \Sigma \mathbb{T}^{n-1}$ be the projection onto the j th factor and let $z_j = r_j^\#(z)$. Clearly $z = z_1 \cdots z_k$ where \cdot is the coproduct in $[\Sigma \mathbb{T}^{n-1}, \text{Conf}(n)]$. Since we can choose the representative of τ_{i_j} to be $1 \cdots 1 \cdot \tau_{i_j} \cdot 1 \cdots 1$, repeating the above reasoning yields

$$\kappa(\widehat{\tau}_{i_j}) = \kappa([1 \cdots 1 \cdot \widehat{\tau_{i_j}} \cdot 1 \cdots 1]) = h^\#(1 \cdots 1 \cdot z_j \cdot 1 \cdots 1) = h^\#(r_j^\#(z_j)) = p^\#(z_j),$$

where the last identity follows from the definitions of h , r_j , and p . But then we know from (4.7) that $\kappa(\widehat{\tau}_{i_j}) = p^\#(t_{n,i_j}(i_j))$, so it follows that the adjoint of z_j is $t_{n,i_j}(i_j)$. Since the adjoint of the coproduct is the loop product, we conclude that the adjoint of z equals $\phi(\tau)$, and

$$\kappa(\widehat{\tau}) = h^\#(z) = p^\#(\phi(\tau)).$$

□

Figure 3 shows the intuition – which originated in Section 5 of [7] – behind the above argument.

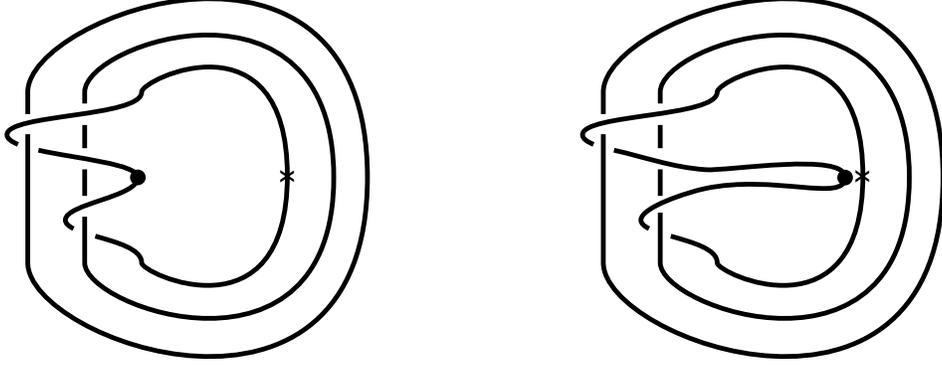


FIGURE 3. The link $\widehat{\tau_{3,2}\tau_{3,1}}$ before and after an isotopy. The point indicated by \bullet on the third component corresponds to the parameter t_1 , while the basepoint of the third component is marked by \times . The isotopy moving \bullet to the basepoint \times makes it clear that the loop in[‡] $\pi_1(\text{Maps}(\mathbb{T}^{n-1}, \text{Conf}(n)), *) \cong T(n)$ corresponding to $\kappa(\widehat{\tau_{3,2}\tau_{3,1}})$ is homotopic to the product of the loops corresponding to $\kappa(\widehat{\tau_{3,2}})$ and $\kappa(\widehat{\tau_{3,1}})$; in other words, that $\kappa(\widehat{\tau_{3,2}\tau_{3,1}}) = p^\#(t_{3,2}(2) \cdot t_{3,1}(1)) = p^\#(\phi(\tau_{3,2}\tau_{3,1}))$.

For (ii) of the Main Theorem, consider any $[L] \in BLM(n)$. Expanding in $\{B(n, \sigma)\}$ we obtain

$$\kappa([L]) = \sum_{\sigma \in \Sigma(2, \dots, n-1)} e(1, \sigma(2), \dots, \sigma(n-1)) B(n, \sigma),$$

where $e(1, \sigma(2), \dots, \sigma(n-1))$ are integer coefficients. We want to show that

$$e(1, \sigma(2), \dots, \sigma(n-1)) = \mu(1, \sigma(2), \dots, \sigma(n-1); n), \quad (4.9)$$

where $\mu(1, \sigma(2), \dots, \sigma(n-1); n)$ are the Milnor invariants of L defined in Section 1.

As a first step, let $z \in B\mathcal{H}(n)$ be such that $\widehat{z} = L$. Thanks to the normal form (2.7), the definition of ϕ , and Diagram 4.6,

$$z = \prod_{\sigma} \tau(n, \sigma)^{e(1, \sigma(2), \dots, \sigma(n-1))}. \quad (4.10)$$

Thus, it suffices to confirm (4.9) for a link diagram obtained by closing up a pure braid representative of z in the above normal form. The identity (4.9) will follow from

$$\mu(1, \xi(2), \dots, \xi(n-1); n)(\widehat{\tau(n, \sigma)}) = \begin{cases} 1 & \text{if } \xi = \sigma \in \Sigma_{n-2} \\ 0 & \text{if } \xi \neq \sigma, \end{cases} \quad (4.11)$$

$$\mu(i_1, i_2, \dots, i_k; n)(\widehat{\tau(n, \sigma)}) = 0 \quad \text{for } k < n-1,$$

and the product formula [27, p. 7]

$$\begin{aligned} \mu(j_1, \dots, j_p; n)(\widehat{z_1 \cdot z_2}) &= \mu(j_1, \dots, j_p; n)(\widehat{z_1}) + \mu(j_1, \dots, j_p; n)(\widehat{z_2}) \\ &\quad + \sum_{k=1}^{n-1} \mu(j_1, \dots, j_k; n)(\widehat{z_1}) \mu(j_{k+1}, \dots, j_p; n)(\widehat{z_2}), \end{aligned} \quad (4.12)$$

[‡]The basepoint $*$ of $\pi_1(\text{Maps}(\mathbb{T}^{n-1}, \text{Conf}(n)), *)$ can be taken (w.l.o.g.) as the null map obtained from the restriction of the link to its trivial first $n-1$ strands.

where $z_1, z_2 \in \mathcal{H}(n)$ and \cdot denotes the product of string links. Indeed, any proper sublink K of $\widehat{z} = L$ is link-homotopically trivial, meaning that any word w_i representing a longitude in the link group $\pi_1(S^3 - K)/\pi_1(S^3 - K)_q$ is trivial. But then, since removing a component doesn't affect the μ invariants not involving the index of that component, this implies that all $\mu(j_1, \dots, j_k; n)(L)$ with $k < n - 1$ vanish. Hence, (4.12) applied to (4.10) yields

$$\mu(j_1, \dots, j_{n-1}; n)(\widehat{z}) = \sum_{\sigma} e(1, \sigma(2), \dots, \sigma(n-1)) \mu(j_1, \dots, j_{n-1}; n)(\tau(n, \sigma)).$$

Substituting $(j_1, \dots, j_{n-1}) = (1, \sigma(2), \dots, \sigma(n-1))$ and applying (4.11), we obtain (4.9) and, therefore, (ii) of the Main Theorem. One corollary is the previously-known fact [29] that the $(n-2)!$ higher linking numbers $\{\mu(1, \sigma(2), \dots, \sigma(n-1); n)\}$ classify $BLM(n)$.

It remains to prove (4.11). A longitude w_n of the n th component of $\tau(n, \sigma)$ can be read off directly from the combed diagram of $\tau(n, \sigma)$ as the following word in the meridians $\{m_j\}$ of the other components:

$$m_{\sigma} = [m_1, m_{\sigma(2)}, \dots, m_{\sigma(n-1)}].$$

Formally, this word is simply obtained from $\tau(n, \sigma)$ by replacing $\tau_{n,j}$ with m_j . It is a general fact (see [25]) that the leading term in the Magnus expansion of any commutator $[m_i, m_j]$ is equal to the commutator $[X_i, X_j] = X_i X_j - X_j X_i$ in the ring $\mathbb{Z}\langle\langle X_1, \dots, X_n \rangle\rangle$ provided $[X_i, X_j] \neq 0$. Therefore, an inductive argument implies that

$$M(w_n) = 1 + [X_1, X_{\sigma(2)}, \dots, X_{\sigma(n-1)}] + (\text{higher-order terms})$$

(notice that $[X_1, X_{\sigma(2)}, \dots, X_{\sigma(n-1)}] \neq 0$ since it involves distinct variables). Again, an elementary induction on n shows that in the expansion of $X_{\sigma} := [X_1, X_{\sigma(2)}, \dots, X_{\sigma(n-1)}]$ in monomials of degree $n-1$, the monomial $X_1 X_{\sigma(2)} \cdots X_{\sigma(n-1)}$ occurs only once; i.e.,

$$\begin{aligned} M(w_n) &= 1 + X_1 X_{\sigma(2)} \cdots X_{\sigma(n-1)} \\ &\quad + (\text{other terms of order } n-1 \text{ not starting with } X_1). \end{aligned}$$

Therefore, (4.11) follows from (1.2) and the definition of the $\bar{\mu}$ -invariants from Section 1.

This ends the proof of the Main Theorem.

Corollary 1.2 is a direct consequence of the fact that the homotopy periods of the Samelson products $\{B(n, \sigma)\}$ in $\pi_{n-1}(\Omega \text{Conf}(n))$ can be obtained from the general methodology of Sullivan's minimal model theory [34], or Chen's iterated integrals theory [13]. Consult [19] for a basic derivation of such an integral in the case of three component links.

APPENDIX A. INDEPENDENCE OF THE BASEPOINTS

Consider the function space of, respectively, free and based n -component link maps

$$\begin{aligned} \text{LMaps}(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3) &= \{L \in \text{Maps}(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3) \mid L_i(S^1) \cap L_j(S^1) = \emptyset, i \neq j\}, \\ \text{LMaps}_0(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3) &= \{L \in \text{LMaps}(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3) \mid L_i(0) = e_i \text{ for each } i\}, \end{aligned}$$

where $\{e_1, e_2, \dots, e_n\}$ is a set of n distinct points in \mathbb{R}^3 which serve as basepoints for each component. Now, the free link homotopy classes and pointed link homotopy classes can be defined as

$$LM(n) = \pi_0(\text{LMaps}(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3)), \quad LM_0(n) = \pi_0(\text{LMaps}_0(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3)).$$

We have an obvious fibration

$$\text{LMaps}_0(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3) \longrightarrow \text{LMaps}(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3) \xrightarrow{ev_0} \text{Conf}(n),$$

given by the evaluation map ev_0 , defined for $L \in \text{LMaps}(S^1 \sqcup \dots \sqcup S^1, \mathbb{R}^3)$ as

$$ev_0(L) = (L_1(0), \dots, L_n(0)).$$

Since both $\pi_0(\text{Conf}(n))$ and $\pi_1(\text{Conf}(n))$ are trivial, the long exact sequence of homotopy groups implies a bijective correspondence between $LM(n)$ and $LM_0(n)$. By an analogous argument, the free homotopy classes $[\mathbb{T}^n, \text{Conf}(n)]$ and pointed homotopy classes $[(\mathbb{T}^n, *), (\text{Conf}(n), *)]$ are equal as sets.

APPENDIX B. PROOF OF LEMMA 3.3

We will first argue that the $\Omega\psi_i^\#$ are defined on the generators $\{t_{k,j}(\ell)\}$ of $TF(n)$ as

$$\Omega\psi_i^\#(t_{k,j}(\ell)) = \begin{cases} 1 & \text{for } k = i \text{ or } j = i \\ t_{k,j}(\ell) & \text{otherwise.} \end{cases} \quad (\text{B.1})$$

Each ψ_i is a fibration, with the fiber having the homotopy type of $\bigvee_{i=1}^{n-1} S^2$, [9, 4]. After looping maps in the fibration diagram we obtain

$$\begin{array}{ccccc} & & \bigvee_{n-1} \Omega S^2 & & \\ & \nearrow^{B_{*,i}} & \downarrow & & \\ S^1 & \xrightarrow{B_{*,i}} & \Omega\text{Conf}(n) & \xrightarrow{\Omega\psi_i} & \Omega\text{Conf}(n-1). \end{array}$$

The free part of $\pi_1(\bigvee_{n-1} \Omega S^2)$, where $\bigvee_{n-1} \Omega S^2$ is the fiber in the above diagram, is generated by $\{B_{i,k}, B_{r,i} \mid 1 \leq k \leq i-1, i+1 \leq r \leq n\}$. Since the fibration ψ_i admits a section we obtain

$$\ker \pi_*(\Omega\psi_i) \cong \pi_*\left(\bigvee_{n-1} \Omega S^2\right) \subset \pi_*\left(\bigvee_{n-1} \Omega S^2\right) \oplus \pi_*(\Omega\text{Conf}(n-1)) \cong \pi_*(\Omega\text{Conf}(n)).$$

From $t_{k,i}(\ell) = j(\ell)(B_{k,i})$ (3.6) and the conclusion that $\{B_{i,k}, B_{r,i}\}$ generates $\ker \pi_*(\Omega\psi_i)$, the formula in (B.1) follows.

Next we aim to show that every $t \in BTF(n)$ is a product of length $n-1$ commutators in the $t_{k,i}(\ell)$ which do not have repeated lower indices $\{k, i\}$.

Similar to the situation in the proof of Lemma 2.2, no nontrivial commutator $[t_{k_1, i_1}(\ell_1), \dots, t_{k_j, i_j}(\ell_j)]$ of length $< n-1$ can be in $BTF(n)$. Indeed, any such commutator has to be in the image under $j_I^\#$, with $I = (\ell_1, \dots, \ell_j)$, $1 \leq \ell_1, \dots, \ell_j \leq n-1$, of the iterated Samelson product $[B_{k_1, i_1}, \dots, B_{k_j, i_j}]$. Thanks to relation (1.6) and Lemma 3.1, all the B_{k_s, i_s} have to have a common first lower index k ; thus, up to sign, $[B_{k_1, i_1}, \dots, B_{k_j, i_j}]$ equals $[B_{k, i_1}, \dots, B_{k, i_j}]$. Furthermore, if $j < n-1$, then we can find r with $1 \leq r \leq n-1$ such that $r \notin \{i_1, \dots, i_j\}$. Then it follows from (B.1) that

$$\Omega\psi_r^\#([t_{k, i_1}(\ell_1), \dots, t_{k, i_j}(\ell_j)]) \neq 1,$$

so this commutator cannot be in $BTF(n)$.

Likewise, no nontrivial commutator $[t_{k, i_1}(\ell_1), \dots, t_{k, i_{n-1}}(\ell_{n-1})]$, (where $(\ell_1, \dots, \ell_{n-1})$ is a permutation of $(1, \dots, n-1)$) with repeated second lower indices (i.e., $i_p = i_q$ for some $p \neq q$) can be in $BTF(n)$, since again we can find r not in the set $\{i_1, \dots, i_{n-1}\}$ and conclude that the commutator is not in $\ker \Omega\psi_r^\#$. Thus, we are left only with the possibility that $t \in BTF(n)$ is the product of

length $n - 1$ commutators without repeated lower indices. Hence, (3.10) follows from (11) and (12) of Lemma 3.2.

It remains to observe that the $t(n, \sigma)$ defined in (3.9) generate $BTF(n)$ or, equivalently, that

$$\{B(n, \sigma) = [B_{n,1}, B_{n,\sigma(2)}, \dots, B_{n,\sigma(n-1)}]\}$$

generates $\ker \pi_*(\prod_i \Omega\psi_i)$. By abuse of notation we use $BTF(n)$ to denote $\ker \pi_*(\prod_i \Omega\psi_i)$ as well. Since we are only concerned with the free part of $\ker \Psi$ it suffices to work rationally and show the following about the $B(n, \sigma)$ s:

- (a) they span $BTF(n) \otimes \mathbb{Q}$,
- (b) they are linearly independent in $\mathcal{L}(\text{Conf}(n)) = \pi_*(\Omega\text{Conf}(n)) \otimes \mathbb{Q}$.

For (a), note that (by an argument analogous to [25, p. 295]) $BTF(n) \otimes \mathbb{Q}$ is spanned by the simple $(n - 1)$ -fold products $b_I := [B_{n,i_1}, B_{n,i_2}, \dots, B_{n,i_{n-1}}]$ with no repeated B_{n,i_k} , thanks to the preceding paragraphs. Then the issue is to get $B_{n,1}$ to the front of each b_I . This is trivial for length 2 commutators, and for length 3 commutators requires at most an application of the Jacobi identity, which up to sign says $[[a, b], c] = [[c, b], a] + [[c, a], b]$. Inductively, suppose (a) holds for commutators of length $\leq n - 2$ and let

$$b_I := [B_{n,i_1}, \dots, B_{n,i_k}, \dots, B_{n,i_{n-1}}]$$

be any length $(n - 1)$ commutator with $i_k = 1$. We need to “shuffle” the elements of b_I to get $B_{n,1}$ to the front. If $k < n - 1$, then the shuffling is easy because $b_I = [a, B_{n,i_{k+1}}, \dots, B_{n,i_{n-1}}]$, where $a = [B_{n,i_1}, \dots, B_{n,1}]$ is a commutator of length $\leq n - 2$. The inductive hypothesis implies that a is a linear combination of products beginning with $B_{n,1}$ and, consequently, bilinearity of the Samelson product implies that b_I is a linear combination of the $B(n, \sigma)$ s.

The only exceptional case is when $B_{n,1}$ occupies the last spot in b_I ; in other words, when $b_I = [B_{n,i_1}, \dots, B_{n,i_{n-2}}, B_{n,1}]$. Then $b_I = [a_{n-3}, B_{n,i_{n-2}}, B_{n,1}]$, where $a_{n-3} = [B_{n,i_1}, \dots, B_{n,i_{n-3}}]$. The Jacobi identity tells us that, up to sign,

$$b_I = [[a_{n-3}, B_{n,i_{n-2}}], B_{n,1}] = [[a_{n-3}, B_{n,1}], B_{n,i_{n-2}}] + [[B_{n,1}, B_{n,i_{n-2}}], a_{n-3}]$$

By the inductive hypothesis the inner part $[a_{n-3}, B_{n,1}]$ of the first term is a linear combination of products beginning with $B_{n,1}$ and therefore bilinearity of $[\ , \]$ implies again that this term is a combination of products beginning with $B_{n,1}$. The second term $[[B_{n,1}, B_{n,i_{n-2}}], a_{n-3}]$ requires repeated applications of the Jacobi identity, which we carry out up to sign. Since $a_{n-3} = [a_{n-4}, B_{n,i_{n-3}}]$, we have

$$\begin{aligned} [[B_{n,1}, B_{n,i_{n-2}}], a_{n-3}] &= [[B_{n,1}, B_{n,i_{n-2}}], [a_{n-4}, B_{n,i_{n-3}}]] \\ &= [[[B_{n,1}, B_{n,i_{n-2}}], a_{n-4}], B_{n,i_{n-3}}] + [[B_{n,1}, B_{n,i_{n-2}}], B_{n,i_{n-3}}], a_{n-4}] \end{aligned}$$

and by induction the first term is in the span of $\{B(n, \sigma)\}$. Thus, applying the Jacobi identity this way $n - 5$ times yields

$$[[B_{n,1}, B_{n,i_{n-2}}], a_{n-3}] = (\text{terms in span}\{B(n, \sigma)\}) + [[B_{n,1}, B_{n,i_{n-2}}, \dots, B_{n,i_2}], a_1].$$

Since $[[B_{n,1}, B_{n,i_{n-2}}, \dots, B_{n,i_2}], a_1] = [B_{n,1}, B_{n,i_{n-2}}, \dots, B_{n,i_2}, B_{n,i_1}]$ is in $\{B(n, \sigma)\}$ we conclude that b_I is in the span of $\{B(n, \sigma)\}$, which implies (a).

Part (b) follows by considering $\mathcal{L}(\text{Conf}(n))$ as a subalgebra of the universal enveloping algebra $U\mathcal{L}(\text{Conf}(n))$. Each $B(n, \sigma)$ is then an element of $U\mathcal{L}(\text{Conf}(n))$ and can be expanded in the tensor

product as

$$\begin{aligned} B(n, \sigma) &= [B_{n,1}, B_{n,\sigma(2)}, \dots, B_{n,\sigma(n-1)}] \\ &= B_{n,1} \otimes B_{n,\sigma(2)} \otimes \dots \otimes B_{n,\sigma(n-1)} + \{\text{other terms not starting with } B_{n,1}\}. \end{aligned}$$

By an elementary induction it follows that in the above tensor product expansion the monomial $B_{n,1} \otimes B_{n,\sigma(2)} \otimes \dots \otimes B_{n,\sigma(n-1)}$ occurs only once. Moreover, the monomials $B_{n,1} \otimes B_{n,\sigma(2)} \otimes \dots \otimes B_{n,\sigma(n-1)}$ are independent in $U\mathcal{L}(\text{Conf}(n))$ by the Poincaré-Birkhoff-Witt Theorem (c.f. e.g. [13]) and the fact that $\{B_{i,j}\}$ form a basis of $\mathcal{L}(\text{Conf}(n))$, see Section 1.2. This in turn implies independence of $\{B(n, \sigma)\}$ in $U\mathcal{L}(\text{Conf}(n))$ and in $\mathcal{L}(\text{Conf}(n))$.

This completes the proof of Lemma 3.3.

APPENDIX C. PROOF OF LEMMA 4.1

Note that in place of $p^\# \circ j_N^\#$ we may prove injectivity of the map

$$p^\# \circ j_n^\# : \pi_n(\text{Conf}(n)) \longrightarrow [\mathbb{T}^n, \text{Conf}(n)],$$

where $j_n^\# : \pi_n(\text{Conf}(n)) \longrightarrow [\Sigma\mathbb{T}^{n-1}, \text{Conf}(n)]$ is the adjoint of $j_N^\#$. We simplify further by considering the projection

$$q : \mathbb{T} \longrightarrow \mathbb{T}/\mathbb{T}^{(n-1)} \cong S^n,$$

where $\mathbb{T}^{(n-1)}$ is the $(n-1)$ -skeleton of $\mathbb{T} := \mathbb{T}^n$, and observing that, up to homotopy equivalence, $q^\# : \pi_n(\text{Conf}(n)) \longrightarrow [\mathbb{T}^n, \text{Conf}(n)]$ satisfies

$$q^\# = p^\# \circ j_n^\#.$$

We wish to think about \mathbb{T} as

$$\mathbb{T} = \mathbb{T}^{(n-1)} \cup_\varphi B^n,$$

where B^n is an embedded n -ball in \mathbb{T} and $\varphi : \partial B^n \rightarrow \mathbb{T}^{(n-1)}$ is the attaching map. We consider the associated cofibration

$$S^{n-1} \cong \partial B^n \xrightarrow{\varphi} \mathbb{T}^{(n-1)} \xrightarrow{i} C_\varphi \cong \mathbb{T},$$

the induced cofibration sequence (where \cong denotes a homotopy equivalence)

$$\partial B^n \xrightarrow{\varphi} \mathbb{T}^{(n-1)} \longrightarrow C_\varphi \cong \mathbb{T} \xrightarrow{q} \Sigma\partial B^n \cong \mathbb{T}/\mathbb{T}^{(n-1)} \xrightarrow{\Sigma\varphi} \Sigma\mathbb{T}^{(n-1)} \longrightarrow \dots,$$

and the exact sequence

$$\begin{aligned} \pi_{n-1}(\text{Conf}(n)) &\xleftarrow{\varphi^\#} [\mathbb{T}^{(n-1)}, \text{Conf}(n)] \xleftarrow{q^\#} [\mathbb{T}, \text{Conf}(n)] \xleftarrow{q^\#} \pi_n(\text{Conf}(n)) \dots \\ &\dots \xleftarrow{\Sigma\varphi^\#} [\Sigma\mathbb{T}^{(n-1)}, \text{Conf}(n)] \xleftarrow{\dots} \end{aligned}$$

(where we used the fact that $\mathbb{T}/\mathbb{T}^{(n-1)} \cong S^n$). The Barratt–Puppe action [5] of $\pi_n(\text{Conf}(n))$ on $[\mathbb{T}, \text{Conf}(n)]$ arises from the map $\mathbb{T} \longrightarrow \mathbb{T} \vee \mathbb{T}/\mathbb{T}^{(n-1)} = \mathbb{T} \vee S^n$ which collapses the boundary of the embedded ball B^n in \mathbb{T} to a point. We indicate this action by

$$\begin{aligned} \circ : \pi_n(\text{Conf}(n)) \times [\mathbb{T}, \text{Conf}(n)] &\longrightarrow [\mathbb{T}, \text{Conf}(n)], \\ a \circ f &= f^a, \quad a \in \pi_n(\text{Conf}(n)), \quad f \in [\mathbb{T}, \text{Conf}(n)]. \end{aligned}$$

One property of this action, which is clear from the definition, is that for any $a, c \in \pi_n(\text{Conf}(n))$ we have

$$a \circ q^\#(c) = q^\#(c)^a = q^\#(a + c).$$

Turning to the proof of the injectivity of $q^\#$, suppose $a, b \in \pi_n(\text{Conf}(n))$ are such that $q^\#(a) = q^\#(b)$. Thanks to the above property of the action,

$$[1] = q^\#(a - a) = q^\#(a)^{-a} = q^\#(b)^{-a} = q^\#(b - a).$$

By exactness of the cofibration sequence it follows that $b - a$ is in the image of the map $\Sigma\varphi^\# : [\Sigma\mathbb{T}^{(n-1)}, \text{Conf}(n)] \rightarrow \pi_n(\text{Conf}(n))$. But $\Sigma\varphi^\#$, as shown below, is trivial and so $a = b$, which implies the claim.

To see that $\Sigma\varphi^\#$ is trivial, note that there exists a map $s : \Sigma\mathbb{T} \rightarrow \Sigma\mathbb{T}^{(n-1)}$ such that the composite

$$\Sigma\mathbb{T}^{(n-1)} \xrightarrow{\subset} \Sigma\mathbb{T} \xrightarrow{s} \Sigma\mathbb{T}^{(n-1)}$$

is homotopic to the identity. Indeed, \mathbb{T} is a product of circles and therefore $\Sigma\mathbb{T}$ is a wedge of spheres (as in (3.5)). Hence it suffices to choose s to be the projection onto those factors of $\Sigma\mathbb{T}$ in the wedge which correspond to $\Sigma\mathbb{T}^{(n-1)}$. Then,

$$\mathbb{T}/\mathbb{T}^{(n-1)} \xrightarrow{\Sigma\varphi} \Sigma\mathbb{T}^{(n-1)} \xrightarrow{\subset} \Sigma\mathbb{T} \xrightarrow{s} \Sigma\mathbb{T}^{(n-1)}$$

is homotopic to $\Sigma\varphi$. But the composition of $\Sigma\varphi$ and \subset is null, which implies the triviality of $\Sigma\varphi^\#$.

This completes the proof of Lemma 4.1.

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