Spherical Geometry and the Least Symmetric Triangle

Supplementary Materials

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Least symmetric obtuse triangle

As explained in the text of the paper, the least symmetric obtuse triangle corresponds to a point on the sphere which must lie on the
curve

$$p(t) = \cos(t)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) + \sin(t)\left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}}\right) = \left(\frac{\sqrt{2}\cos(t)+\sin(t)}{2}, \frac{\sqrt{2}\cos(t)-\sin(t)}{2}, \frac{\sin(t)}{\sqrt{2}}\right).$$

```
p[t_] := {(Sqrt[2] Cos[t] + Sin[t]) / 2, (Sqrt[2] Cos[t] - Sin[t]) / 2, Sin[t] / Sqrt[2]};
```

On the other hand, the right triangles lie on the curve

$$q(x) = \left(x, \sqrt{\frac{1-x^2}{1+x^2}}, x\sqrt{\frac{1-x^2}{1+x^2}}\right)$$

```
q[x_] := {x, Sqrt[(1 - x^2) / (1 + x^2)], x Sqrt[(1 - x^2) / (1 + x^2)]};
```

We see that the distance from $p(t)$ to $q(x)$ is

```
ArcCos[p[t].q[x]]
```

$$\text{ArcCos}\left[\frac{1}{2}\sqrt{\frac{1-x^2}{1+x^2}}\left(\sqrt{2}\,\text{Cos}[t] - \text{Sin}[t]\right) + \frac{x\sqrt{\frac{1-x^2}{1+x^2}}\,\text{Sin}[t]}{\sqrt{2}} + \frac{1}{2}x\left(\sqrt{2}\,\text{Cos}[t] + \text{Sin}[t]\right)\right]$$

and the challenge is to determine when this quantity equals $\arcsin\left(\frac{\sin(t)}{\sqrt{2}}\right) = \arccos\left(\frac{\sqrt{3+\cos(2s)}}{2}\right)$. After eliminating the inverse cosines, we are solving

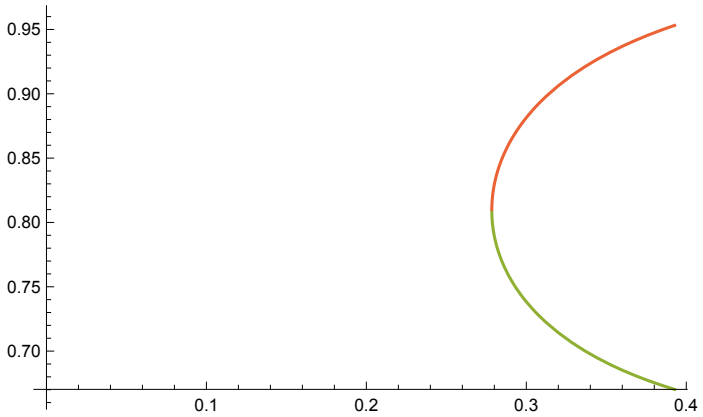
$$\text{sols} = \text{Solve}\left[\frac{1}{2}\sqrt{\frac{1-x^2}{1+x^2}}\left(\sqrt{2}\,\text{Cos}[t]-\text{Sin}[t]\right)+\frac{x\sqrt{\frac{1-x^2}{1+x^2}}\,\text{Sin}[t]}{\sqrt{2}}+\frac{1}{2}x\left(\sqrt{2}\,\text{Cos}[t]+\text{Sin}[t]\right)==\frac{\sqrt{3+\text{Cos}[2t]}}{2},\,x\right]$$

$$\left\{\left\{x\to -\left(\left(-2\sqrt{2}\,\text{Cos}[t]\sqrt{3+\text{Cos}[2t]}+4\,\text{Cos}[t]\,\text{Sin}[t]-2\sqrt{3+\text{Cos}[2t]}\,\text{Sin}[t]-2\sqrt{2}\,\text{Sin}[t]^2\right)\Big/\left(4\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)\right)\right)-\frac{1}{2}\sqrt{\frac{6\,\text{Cos}[t]^2}{(\cdots 1\cdots)^2}+\frac{2(\cdots 1\cdots)^2\,\text{Cos}[2t]}{(\cdots 1\cdots)^2}+\cdots 29\cdots+\frac{(\cdots 1\cdots)^{1/3}}{3\times 2^{1/3}}}-\frac{1}{2}\sqrt{\left(\cdots 42\cdots+\frac{\cdots 1\cdots}{\cdots 1\cdots}-\frac{(\cdots 1\cdots)^{1/3}}{3\times 2^{1/3}}-\left(-\left(\left(8\left(-2\sqrt{2}\,\text{Cos}[t]\sqrt{3+\text{Cos}[2t]}-4\,\text{Cos}[t]\,\text{Sin}[t]-\cdots 1\cdots+2\sqrt{2}\,\text{Sin}[t]^2\right)\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)\right)+\frac{(\cdots 1\cdots)(\cdots 1\cdots)}{2(\cdots 1\cdots)^2+(\cdots 1\cdots)+3(\cdots 1\cdots)}\right)\Big/\left(4\sqrt{\left(\frac{6\,\text{Cos}[t]^2}{(2(\cdots 1\cdots)^2+(\cdots 1\cdots)+3(\cdots 1\cdots))^2}+\frac{2\,\text{Cos}[t]^2\,\text{Cos}[2t]}{(\cdots 1\cdots)^2}+\cdots 29\cdots+\frac{(\cdots 1\cdots)^{1/3}}{3\times 2^{1/3}}\right)}\right)\right)\right\},\ \cdots 2\cdots,\ \left\{x\to -\frac{\cdots 1\cdots}{4(\cdots 1\cdots)}+\cdots 1\cdots+\cdots 1\cdots\right\}\right\}$$

large output | show less | show more | show all | set size limit...

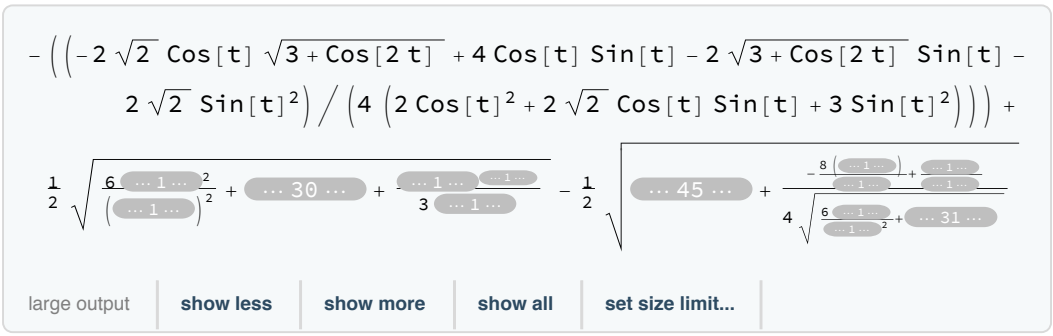
There are four solutions above, but two are always complex and two are sometimes complex and sometimes real. Now, we're after the smallest value of *t* that gives a possible solution, which is to say, a real value of *x*, so this will be the value of *t* where two (conjugate) complex solutions become real and then bifurcate, which we see happens in the plot at a value of *t* slightly smaller than 0.3.

```
Plot[Evaluate[x /. # & /@ sols], {t, 0, π/8}]
```



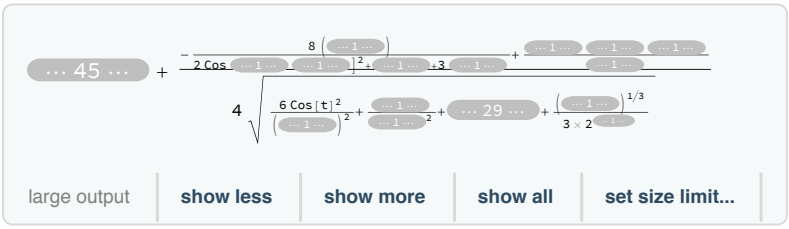
Based on *Mathematica*'s default color scheme, we can see that these are the third and fourth solutions found above, so the value of *t* we want is the one where the following function switches from complex to real:

```
x /. sols[[3]]
```



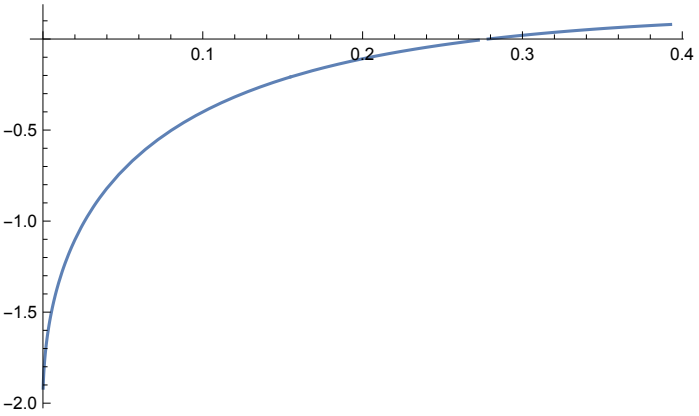
The above is the sum of three functions: a fraction and two square roots. The only term that is problematic is the last square root: the term under the square root can be negative. So we extract the term under the last square root:

```
complexterm = %[[3, 2, 1]]
```



The value of *t* that we want is the root of this expression:

```
Plot[complexterm, {t, 0, π/8}]
```



There are (by *Mathematica*'s count) 28 terms in this expression:

```
Length[complexterm]
```

```
28
```

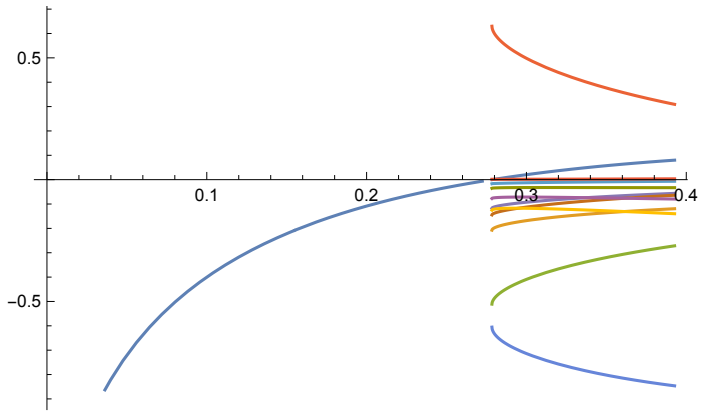
And, of those 28, terms 17–27 can be complex, which we can verify by evaluating them at a value of *t* smaller then the value where they become real; say at $t = 0.1$.

```
TableForm[Table[{i, Chop[complexterm[[i]] /. t → 0.2]}, {i, 1, 28, 1}]]
```

1	−0.859012
2	−0.263734
3	−0.246257
4	−0.075606
5	0.318607
6	−0.017649
7	−0.0226032
8	−0.0054186
9	0.00647977
10	−0.006546
11	−0.000464398
12	0.772144
13	0.988891
14	0.237064
15	−5.9943
16	3.51661
17	−0.232023 + 0.0925673 i
18	−0.594307 + 0.237103 i
19	0.761133 − 0.30366 i
20	−0.142472 + 0.0568401 i
21	−0.182464 + 0.0727955 i
22	−0.0218708 + 0.00872553 i
23	−0.0732627 + 0.0292287 i
24	−0.046914 + 0.0187167 i
25	−0.0224931 + 0.0089738 i
26	0.00048194 − 0.000192274 i
27	−0.554191 − 0.221099 i
28	2.65178

Each of these potentially-complex terms switches from complex to real at the same time the overall function switches from negative to positive.

```
Plot[
 Evaluate@Prepend[Table[complexterm[[i]], {i, 17, 27}], complexterm], {t, 0, π/8}]
```



So we can focus on a single term to see when it stops being complex:

```
complexterm[[17]]
```

$$-\left(\left(3\times 2^{1/3}\right)\Big/\left(\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^2\right.\right.$$

$$\left(\frac{54}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} + \right.$$

$$\frac{216\,\text{Cos}[t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} -$$

$$\frac{1440\,\text{Cos}[t]^4}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{1280\,\text{Cos}[t]^6}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{54\,\text{Cos}[2\,t]}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{144\,\text{Cos}[t]^2\,\text{Cos}[2\,t]}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} -$$

$$\frac{480\,\text{Cos}[t]^4\,\text{Cos}[2\,t]}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{18\,\text{Cos}[2\,t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{24\,\text{Cos}[t]^2\,\text{Cos}[2\,t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{2\,\text{Cos}[2\,t]^3}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} -$$

$$\frac{1296\sqrt{2}\,\text{Cos}[t]^2\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{1728\sqrt{2}\,\text{Cos}[t]^4\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} -$$

$$\frac{432\sqrt{2}\,\text{Cos}[t]^2\,\text{Cos}[2\,t]\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} -$$

$$\frac{1296\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{5184\,\text{Cos}[t]^2\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{576\,\text{Cos}[t]^4\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} -$$

$$\frac{864\,\text{Cos}[2\,t]\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$\frac{1728\,\text{Cos}[t]^2\,\text{Cos}[2\,t]\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} -$$

$$\frac{144\,\text{Cos}[2\,t]^2\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2 + 2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t] + 3\,\text{Sin}[t]^2\right)^3} +$$

$$
\frac{648\sqrt{2}\ \sqrt{3+\cos[2t]}\ \sin[t]^3}{\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3}+
\frac{864\sqrt{2}\ \cos[t]^2\sqrt{3+\cos[2t]}\ \sin[t]^3}{\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3}+
\frac{216\sqrt{2}\ \cos[2t]\ \sqrt{3+\cos[2t]}\ \sin[t]^3}{\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3}+
\frac{2160\sin[t]^4}{\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3}-
\frac{1440\cos[t]^2\sin[t]^4}{\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3}+
\frac{720\cos[2t]\ \sin[t]^4}{\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3}-
\frac{864\sqrt{2}\ \sqrt{3+\cos[2t]}\ \sin[t]^5}{\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3}+
\frac{432\sin[t]^6}{\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3}+
$$

$$
\sqrt{\left(\left(4\left(-9-24\cos[t]^2+32\cos[t]^4-6\cos[2t]-8\cos[t]^2\cos[2t]-\cos[2t]^2-72\sin[t]^2-48\cos[t]^2\sin[t]^2-24\cos[2t]\sin[t]^2+12\sin[t]^4\right)^3\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^6+\left(54\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+\left(216\cos[t]^2\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3-\left(1440\cos[t]^4\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+\left(1280\cos[t]^6\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+\left(54\cos[2t]\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+\left(144\cos[t]^2\cos[2t]\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3-\left(480\cos[t]^4\cos[2t]\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+\left(18\cos[2t]^2\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+\left(24\cos[t]^2\cos[2t]^2\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+\left(2\cos[2t]^3\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3-\left(1296\sqrt{2}\ \cos[t]^2\sqrt{3+\cos[2t]}\ \sin[t]\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+\left(1728\sqrt{2}\ \cos[t]^4\sqrt{3+\cos[2t]}\ \sin[t]\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3-\left(432\sqrt{2}\ \cos[t]^2\cos[2t]\sqrt{3+\cos[2t]}\ \sin[t]\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3-\left(1296\sin[t]^2\right)\Big/\left(2\cos[t]^2+2\sqrt{2}\ \cos[t]\ \sin[t]+3\sin[t]^2\right)^3+
$$

$$\left(5184\,\text{Cos}[t]^2\,\text{Sin}[t]^2\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3+\left(576\,\text{Cos}[t]^4\,\text{Sin}[t]^2\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3-\left(864\,\text{Cos}[2t]\,\text{Sin}[t]^2\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3+\left(1728\,\text{Cos}[t]^2\,\text{Cos}[2t]\,\text{Sin}[t]^2\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3-\left(144\,\text{Cos}[2t]^2\,\text{Sin}[t]^2\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3+\left(648\sqrt{2}\,\sqrt{3+\text{Cos}[2t]}\,\text{Sin}[t]^3\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3+\left(864\sqrt{2}\,\text{Cos}[t]^2\sqrt{3+\text{Cos}[2t]}\,\text{Sin}[t]^3\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3+\left(216\sqrt{2}\,\text{Cos}[2t]\sqrt{3+\text{Cos}[2t]}\,\text{Sin}[t]^3\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3+\left(2160\,\text{Sin}[t]^4\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3-\left(1440\,\text{Cos}[t]^2\,\text{Sin}[t]^4\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3+\left(720\,\text{Cos}[2t]\,\text{Sin}[t]^4\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3-\left(864\sqrt{2}\,\sqrt{3+\text{Cos}[2t]}\,\text{Sin}[t]^5\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3+\left(432\,\text{Sin}[t]^6\right)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3\Big)^2\Big)\Big)^{1/3}\Big)\Big)$$

The problem can only come from the expression under the square root, namely

badterm =

$$\text{FullSimplify}\Big[\Big(\Big(4\left(-9-24\,\text{Cos}[t]^2+32\,\text{Cos}[t]^4-6\,\text{Cos}[2t]-8\,\text{Cos}[t]^2\,\text{Cos}[2t]-\text{Cos}[2t]^2-72\,\text{Sin}[t]^2-48\,\text{Cos}[t]^2\,\text{Sin}[t]^2-24\,\text{Cos}[2t]\,\text{Sin}[t]^2+12\,\text{Sin}[t]^4\right)^3\Big)\Big/\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^6+\Big(\frac{54}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{216\,\text{Cos}[t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{1440\,\text{Cos}[t]^4}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{1280\,\text{Cos}[t]^6}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{54\,\text{Cos}[2t]}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+$$

$$\frac{144\,\text{Cos}[t]^2\,\text{Cos}[2\,t]}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{480\,\text{Cos}[t]^4\,\text{Cos}[2\,t]}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{18\,\text{Cos}[2\,t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{24\,\text{Cos}[t]^2\,\text{Cos}[2\,t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{2\,\text{Cos}[2\,t]^3}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{1296\sqrt{2}\,\text{Cos}[t]^2\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{1728\sqrt{2}\,\text{Cos}[t]^4\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{432\sqrt{2}\,\text{Cos}[t]^2\,\text{Cos}[2\,t]\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{1296\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{5184\,\text{Cos}[t]^2\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{576\,\text{Cos}[t]^4\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{864\,\text{Cos}[2\,t]\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{1728\,\text{Cos}[t]^2\,\text{Cos}[2\,t]\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{144\,\text{Cos}[2\,t]^2\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{648\sqrt{2}\,\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]^3}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{864\sqrt{2}\,\text{Cos}[t]^2\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]^3}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{216\sqrt{2}\,\text{Cos}[2\,t]\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]^3}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+$$

$$\frac{2160\,\text{Sin}[t]^4}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{1440\,\text{Cos}[t]^2\,\text{Sin}[t]^4}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{720\,\text{Cos}[2\,t]\,\text{Sin}[t]^4}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}-\frac{864\sqrt{2}\,\sqrt{3+\text{Cos}[2\,t]}\,\text{Sin}[t]^5}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}+\frac{432\,\text{Sin}[t]^6}{\left(2\,\text{Cos}[t]^2+2\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]+3\,\text{Sin}[t]^2\right)^3}\Bigg)^2\Bigg)\Bigg]$$

$$-\Big(\Big(3456\,\text{Sin}[t]^2\left(22\,830\,\text{Cos}[2\,t]+6392\,\text{Cos}[4\,t]+13\,595\,\text{Cos}[6\,t]+3170\,\text{Cos}[8\,t]+439\,\text{Cos}[10\,t]+2\left(-6829+4\sqrt{2}\,\sqrt{3+\text{Cos}[2\,t]}\,\left(845\,\text{Sin}[t]-1437\,\text{Sin}[3\,t]+423\,\text{Sin}[5\,t]+162\,\text{Sin}[7\,t]+81\,\text{Sin}[9\,t]\right)\right)\right)\Big)\Big/\left(-5+\text{Cos}[2\,t]-2\sqrt{2}\,\text{Sin}[2\,t]\right)^6\Big)$$

So now we solve for when this expression vanishes.

```
Solve[badterm == 0 && 0 < t < 1/2, t, Reals]
```

$$\left\{\left\{t\to 2\,\text{ArcTan}\left[\text{Root}\left[16-992\,\#1^2+9689\,\#1^4-36\,232\,\#1^6+100\,908\,\#1^8-197\,080\,\#1^{10}+238\,166\,\#1^{12}-197\,080\,\#1^{14}+100\,908\,\#1^{16}-36\,232\,\#1^{18}+9689\,\#1^{20}-992\,\#1^{22}+16\,\#1^{24}\,\&,\,7\right]\right]\right\}\right\}$$

So, as claimed in the paper, the correct value of t is $t_0 = 2\arctan(\alpha)$, where α is the seventh root of the polynomial

$$16-992\,\#1^2+9689\,\#1^4-36\,232\,\#1^6+100\,908\,\#1^8-197\,080\,\#1^{10}+238\,166\,\#1^{12}-197\,080\,\#1^{14}+100\,908\,\#1^{16}-36\,232\,\#1^{18}+9689\,\#1^{20}-992\,\#1^{22}+16\,\#1^{24}\,\&[z]$$

$$16-992\,z^2+9689\,z^4-36\,232\,z^6+100\,908\,z^8-197\,080\,z^{10}+238\,166\,z^{12}-197\,080\,z^{14}+100\,908\,z^{16}-36\,232\,z^{18}+9689\,z^{20}-992\,z^{22}+16\,z^{24}$$

and we see that the seventh root is the smallest positive root:

```
TableForm@Table[{i, N[
    Root[16 - 992 #1^2 + 9689 #1^4 - 36 232 #1^6 + 100 908 #1^8 - 197 080 #1^10 + 238 166 #1^12 - 197 080
        #1^14 + 100 908 #1^16 - 36 232 #1^18 + 9689 #1^20 - 992 #1^22 + 16 #1^24 &, i]]}, {i, 1, 24}]
```

1	-7.13714
2	-2.61199
3	-1.35783
4	-0.736468
5	-0.38285
6	-0.140112
7	0.140112
8	0.38285
9	0.736468
10	1.35783
11	2.61199
12	7.13714
13	-1.1465 - 1.02639 i
14	-1.1465 + 1.02639 i
15	-0.876274 - 0.481812 i
16	-0.876274 + 0.481812 i
17	-0.484174 - 0.433454 i
18	-0.484174 + 0.433454 i
19	0.484174 - 0.433454 i
20	0.484174 + 0.433454 i
21	0.876274 - 0.481812 i
22	0.876274 + 0.481812 i
23	1.1465 - 1.02639 i
24	1.1465 + 1.02639 i

Least symmetric acute triangle

The same strategy works for finding the least symmetric acute triangle. The points of interest are those on the curve

$$\tilde{p}(t) = \cos(t)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + \sin(t)\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

```
p2[t_] := Cos[t] 1 / Sqrt[3] {1, 1, 1} + Sin[t] 1 / Sqrt[2] {1, 0, -1};
```

and the distance from $\tilde{p}(t)$ to $q(x)$ is

```
ArcCos[p2[t].q[x]]
```

$$\text{ArcCos}\left[\frac{\sqrt{\frac{1-x^2}{1+x^2}}\,\text{Cos}[t]}{\sqrt{3}} + x\sqrt{\frac{1-x^2}{1+x^2}}\left(\frac{\text{Cos}[t]}{\sqrt{3}} - \frac{\text{Sin}[t]}{\sqrt{2}}\right) + x\left(\frac{\text{Cos}[t]}{\sqrt{3}} + \frac{\text{Sin}[t]}{\sqrt{2}}\right)\right]$$

whereas the distance from $\tilde{p}(t)$ to either of the bounding great circles is $\arcsin(\frac{\sin(t)}{2})$. So we solve for those values of x for which these two distances are equal:

```
sols2 =
```

$$\text{Solve}\left[\text{ArcCos}\left[\frac{\sqrt{\frac{1-x^2}{1+x^2}}\,\text{Cos}[t]}{\sqrt{3}} + x\sqrt{\frac{1-x^2}{1+x^2}}\left(\frac{\text{Cos}[t]}{\sqrt{3}} - \frac{\text{Sin}[t]}{\sqrt{2}}\right) + x\left(\frac{\text{Cos}[t]}{\sqrt{3}} + \frac{\text{Sin}[t]}{\sqrt{2}}\right)\right] == \text{ArcSin}[\text{Sin}[t]/2], x\right]$$

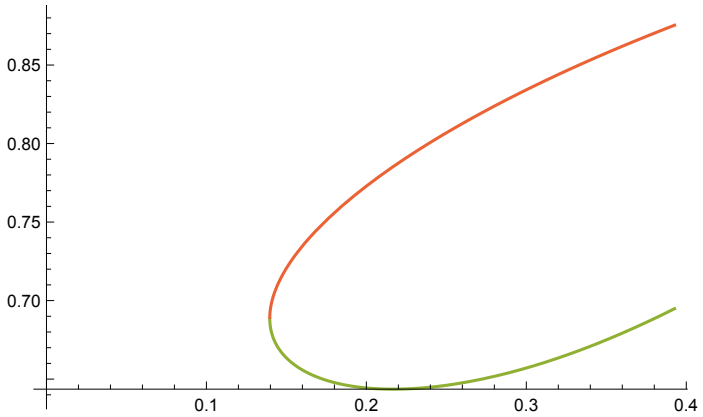
$$\left\{\left\{x \to -\left(\left(2\,\text{Cos}[t]^2 - \sqrt{6}\,\text{Cos}[t]\,\text{Sin}[t] - 2\sqrt{3}\,\text{Cos}[t]\sqrt{1 - \frac{\cdots 1 \cdots^2}{4}} - 3\sqrt{2}\,\text{Sin}[t]\sqrt{1 - \frac{\text{Sin}[t]^2}{4}}\right) \Big/ \left(4\left(2\,\text{Cos}[t]^2 + 3\,\text{Sin}[t]^2\right)\right)\right) - \frac{1}{2}\sqrt{\left(\frac{\text{Cos}[t]^4}{\left(\cdots 1 \cdots\right)^2} - \frac{\sqrt{6}\left(\cdots 1 \cdots\right)^3\text{Sin}[t]}{\left(\cdots 1 \cdots\right)^2} + \cdots 23 \cdots + \frac{\cdots 1 \cdots}{\cdots 1 \cdots} + \frac{\left(\cdots 1 \cdots\right)^{1/3}}{3 \times 2^{1/3}}\right) - \frac{1}{2}\sqrt{-\frac{\text{Cos}[t]^4}{\left(2\left(\cdots 1 \cdots\right)^2 + \cdots 1 \cdots\right)^2} + \cdots 35 \cdots}}\right\},$$

$$\left\{x \to \cdots 1 \cdots\right\}, \cdots 1 \cdots, \left\{x \to \cdots 1 \cdots\right\}\Big\}$$

large output | show less | show more | show all | set size limit...

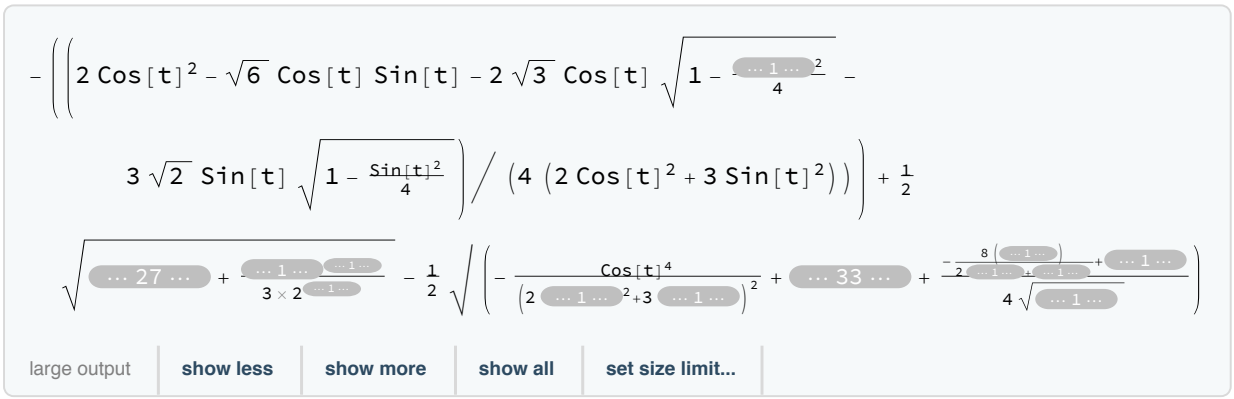
Again, we're looking for the smallest value of *t* that yields a real solution

```
Plot[Evaluate[x /. # & /@ sols2], {t, 0, π/8}]
```



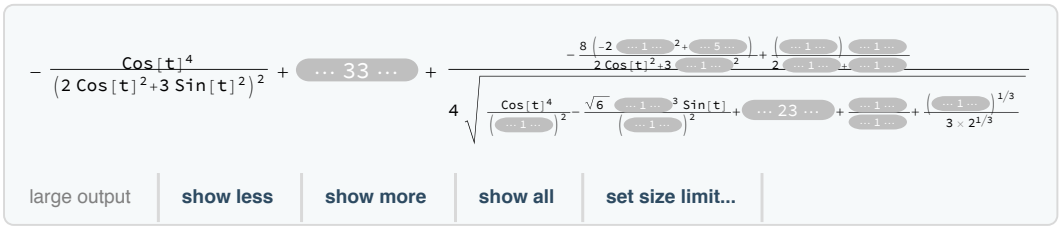
...which is now slightly larger than 0.1. Again, the third and fourth solutions switch from complex to real at this point, so we focus on the third solution.

```
x /. sols2[[3]]
```



Just as before, the trouble is under the last square root:

```
complexterm2 = %[[3, 2, 1]]
```



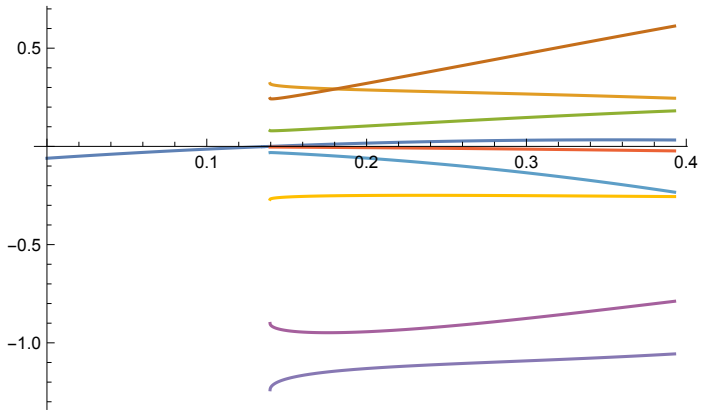
and some terms are real and some aren't:

```
TableForm@Table[{i, complexterm2[[i]] /. t → .1}, {i, 1, Length[complexterm2]}]
```

1	−0.242617
2	0.0596278
3	−0.00366366
4	0.843618
5	−0.0127391
6	−0.733348
7	−0.180234
8	−0.011074
9	0.328375
10	−0.0807042
11	0.242113
12	0.992562
13	0.841289
14	−4.44704
15	0.311916 − 0.0331295 i
16	0.0557522 − 0.00592159 i
17	−0.00171277 + 0.000181918 i
18	−1.19995 + 0.127449 i
19	0.168519 − 0.0178989 i
20	−0.0155313 + 0.00164962 i
21	−0.259073 + 0.0275168 i
22	−0.940074 − 0.0998478 i
23	$4.26999 + 6.55285 \times 10^{-17}$ i

Again, the complex terms become real at the same moment the expression changes from negative to positive; since this expression is under a square root, this is when the whole expression changes from complex to real.

```
Plot[
 Evaluate@Prepend[Table[complexterm2[[i]], {i, 15, 22}], complexterm2], {t, 0, π/8}]
```



Again, the trouble is the expression under the square root in any of these terms:

```
complexterm2[[15]]
```

$$\left(11 \times 2^{1/3} \text{Cos}[t]^4\right) \Big/ \left(3 \left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^2 \left(\frac{290 \text{Cos}[t]^6}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{228 \sqrt{6}\ \text{Cos}[t]^5 \text{Sin}[t]}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \right.\right.$$

$$\frac{468 \text{Cos}[t]^4 \text{Sin}[t]^2}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{312 \sqrt{6}\ \text{Cos}[t]^3 \text{Sin}[t]^3}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{486 \text{Cos}[t]^2 \text{Sin}[t]^4}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} +$$

$$\frac{648 \sqrt{3}\ \text{Cos}[t]^5 \sqrt{1 - \frac{\text{Sin}[t]^2}{4}}}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} - \frac{324 \sqrt{3}\ \text{Cos}[t]^3 \text{Sin}[t]^2 \sqrt{1 - \frac{\text{Sin}[t]^2}{4}}}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} -$$

$$\frac{972 \sqrt{3}\ \text{Cos}[t]\ \text{Sin}[t]^4 \sqrt{1 - \frac{\text{Sin}[t]^2}{4}}}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{666 \text{Cos}[t]^4 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} -$$

$$\frac{1224 \sqrt{6}\ \text{Cos}[t]^3 \text{Sin}[t] \left(1 - \frac{\text{Sin}[t]^2}{4}\right)}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{1080 \text{Cos}[t]^2 \text{Sin}[t]^2 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} -$$

$$\frac{648 \sqrt{6}\ \text{Cos}[t]\ \text{Sin}[t]^3 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{1458 \text{Sin}[t]^4 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} -$$

$$\frac{648 \sqrt{3}\ \text{Cos}[t]^3 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)^{3/2}}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{972 \sqrt{3}\ \text{Cos}[t]\ \text{Sin}[t]^2 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)^{3/2}}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} -$$

$$\frac{594 \text{Cos}[t]^2 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)^2}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{756 \sqrt{6}\ \text{Cos}[t]\ \text{Sin}[t] \left(1 - \frac{\text{Sin}[t]^2}{4}\right)^2}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} -$$

$$\left.\left.\frac{972 \text{Sin}[t]^2 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)^2}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \frac{54 \left(1 - \frac{\text{Sin}[t]^2}{4}\right)^3}{\left(2 \text{Cos}[t]^2 + 3 \text{Sin}[t]^2\right)^3} + \right.\right)$$

$$\sqrt{\left(\left(4\left(11\,\text{Cos}[t]^4+8\sqrt{6}\,\text{Cos}[t]^3\,\text{Sin}[t]-6\,\text{Cos}[t]^2\,\text{Sin}[t]^2-42\,\text{Cos}[t]^2\left(1-\frac{\text{Sin}[t]^2}{4}\right)+24\sqrt{6}\,\text{Cos}[t]\,\text{Sin}[t]\left(1-\frac{\text{Sin}[t]^2}{4}\right)-54\,\text{Sin}[t]^2\left(1-\frac{\text{Sin}[t]^2}{4}\right)-9\left(1-\frac{\text{Sin}[t]^2}{4}\right)^2\right)^3\right)\Big/\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^6+\left(\frac{290\,\text{Cos}[t]^6}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{228\sqrt{6}\,\text{Cos}[t]^5\,\text{Sin}[t]}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{468\,\text{Cos}[t]^4\,\text{Sin}[t]^2}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{312\sqrt{6}\,\text{Cos}[t]^3\,\text{Sin}[t]^3}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{486\,\text{Cos}[t]^2\,\text{Sin}[t]^4}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{648\sqrt{3}\,\text{Cos}[t]^5\sqrt{1-\frac{\text{Sin}[t]^2}{4}}}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}-\frac{324\sqrt{3}\,\text{Cos}[t]^3\,\text{Sin}[t]^2\sqrt{1-\frac{\text{Sin}[t]^2}{4}}}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}-\frac{972\sqrt{3}\,\text{Cos}[t]\,\text{Sin}[t]^4\sqrt{1-\frac{\text{Sin}[t]^2}{4}}}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{666\,\text{Cos}[t]^4\left(1-\frac{\text{Sin}[t]^2}{4}\right)}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}-\frac{1224\sqrt{6}\,\text{Cos}[t]^3\,\text{Sin}[t]\left(1-\frac{\text{Sin}[t]^2}{4}\right)}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{1080\,\text{Cos}[t]^2\,\text{Sin}[t]^2\left(1-\frac{\text{Sin}[t]^2}{4}\right)}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}-\frac{648\sqrt{6}\,\text{Cos}[t]\,\text{Sin}[t]^3\left(1-\frac{\text{Sin}[t]^2}{4}\right)}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{1458\,\text{Sin}[t]^4\left(1-\frac{\text{Sin}[t]^2}{4}\right)}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}-\frac{648\sqrt{3}\,\text{Cos}[t]^3\left(1-\frac{\text{Sin}[t]^2}{4}\right)^{3/2}}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{972\sqrt{3}\,\text{Cos}[t]\,\text{Sin}[t]^2\left(1-\frac{\text{Sin}[t]^2}{4}\right)^{3/2}}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}-\frac{594\,\text{Cos}[t]^2\left(1-\frac{\text{Sin}[t]^2}{4}\right)^2}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{756\sqrt{6}\,\text{Cos}[t]\,\text{Sin}[t]\left(1-\frac{\text{Sin}[t]^2}{4}\right)^2}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}-\frac{972\,\text{Sin}[t]^2\left(1-\frac{\text{Sin}[t]^2}{4}\right)^2}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}+\frac{54\left(1-\frac{\text{Sin}[t]^2}{4}\right)^3}{\left(2\,\text{Cos}[t]^2+3\,\text{Sin}[t]^2\right)^3}\right)^2\Bigg)\Bigg)^{1/3}\Bigg)}$$

```
badterm2 = FullSimplify[
```

$$\left(\left(4\left(11\cos[t]^4+8\sqrt{6}\,\cos[t]^3\sin[t]-6\cos[t]^2\sin[t]^2-42\cos[t]^2\left(1-\frac{\sin[t]^2}{4}\right)+\right.\right.\right.$$

$$\left.24\sqrt{6}\,\cos[t]\sin[t]\left(1-\frac{\sin[t]^2}{4}\right)-54\sin[t]^2\left(1-\frac{\sin[t]^2}{4}\right)-\right.$$

$$\left.\left.9\left(1-\frac{\sin[t]^2}{4}\right)^2\right)^3\right)\Big/\left(2\cos[t]^2+3\sin[t]^2\right)^6+$$

$$\left(\frac{290\cos[t]^6}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{228\sqrt{6}\,\cos[t]^5\sin[t]}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{468\cos[t]^4\sin[t]^2}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\right.$$

$$\frac{312\sqrt{6}\,\cos[t]^3\sin[t]^3}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{486\cos[t]^2\sin[t]^4}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+$$

$$\frac{648\sqrt{3}\,\cos[t]^5\sqrt{1-\frac{\sin[t]^2}{4}}}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}-\frac{324\sqrt{3}\,\cos[t]^3\sin[t]^2\sqrt{1-\frac{\sin[t]^2}{4}}}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}-$$

$$\frac{972\sqrt{3}\,\cos[t]\sin[t]^4\sqrt{1-\frac{\sin[t]^2}{4}}}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{666\cos[t]^4\left(1-\frac{\sin[t]^2}{4}\right)}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}-$$

$$\frac{1224\sqrt{6}\,\cos[t]^3\sin[t]\left(1-\frac{\sin[t]^2}{4}\right)}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{1080\cos[t]^2\sin[t]^2\left(1-\frac{\sin[t]^2}{4}\right)}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}-$$

$$\frac{648\sqrt{6}\,\cos[t]\sin[t]^3\left(1-\frac{\sin[t]^2}{4}\right)}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{1458\sin[t]^4\left(1-\frac{\sin[t]^2}{4}\right)}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}-$$

$$\frac{648\sqrt{3}\,\cos[t]^3\left(1-\frac{\sin[t]^2}{4}\right)^{3/2}}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{972\sqrt{3}\,\cos[t]\sin[t]^2\left(1-\frac{\sin[t]^2}{4}\right)^{3/2}}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}-$$

$$\frac{594\cos[t]^2\left(1-\frac{\sin[t]^2}{4}\right)^2}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{756\sqrt{6}\,\cos[t]\sin[t]\left(1-\frac{\sin[t]^2}{4}\right)^2}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}-$$

$$\left.\left.\frac{972\sin[t]^2\left(1-\frac{\sin[t]^2}{4}\right)^2}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}+\frac{54\left(1-\frac{\sin[t]^2}{4}\right)^3}{\left(2\cos[t]^2+3\sin[t]^2\right)^3}\right)^2\right)\Bigg]$$

$$\frac{1}{16\,384\,\left(-5+\text{Cos}[2\,t]\right)^6}\left(2\left(-5787+356\,\text{Cos}[2\,t]+311\,\text{Cos}[4\,t]+1600\sqrt{6}\,\text{Sin}[2\,t]+224\sqrt{6}\,\text{Sin}[4\,t]\right)^3+\left(402\,346-97\,599\,\text{Cos}[2\,t]+102\,246\,\text{Cos}[4\,t]+18\,991\,\text{Cos}[6\,t]+5184\sqrt{6}\,\sqrt{7+\text{Cos}[2\,t]}\,\left(3\,\text{Cos}[t]+5\,\text{Cos}[3\,t]\right)\text{Sin}[t]^2-59\,640\sqrt{6}\,\text{Sin}[2\,t]-22\,944\sqrt{6}\,\text{Sin}[4\,t]-5784\sqrt{6}\,\text{Sin}[6\,t]\right)^2\right)$$

Given sufficient time, *Mathematica* will find the desired root

```
TimeConstrained[Solve[badterm2 == 0 && 0 < t < 1 / 4, t, Reals], 3600]
```

$$\{\{t\to 2\,\text{ArcTan}\left[\text{Root}\left[131\,072-30\,081\,024\,\#1^2+715\,784\,192\,\#1^4-10\,181\,738\,496\,\#1^6+83\,609\,604\,096\,\#1^8-443\,259\,328\,512\,\#1^{10}+1\,410\,471\,953\,408\,\#1^{12}-1\,858\,643\,071\,488\,\#1^{14}+18\,137\,673\,285\,920\,\#1^{16}-14\,367\,112\,128\,688\,\#1^{18}+56\,162\,265\,469\,488\,\#1^{20}-73\,041\,229\,883\,512\,\#1^{22}+73\,382\,345\,772\,378\,\#1^{24}-122\,601\,623\,733\,111\,\#1^{26}+73\,382\,345\,772\,378\,\#1^{28}-73\,041\,229\,883\,512\,\#1^{30}+56\,162\,265\,469\,488\,\#1^{32}-14\,367\,112\,128\,688\,\#1^{34}+18\,137\,673\,285\,920\,\#1^{36}-1\,858\,643\,071\,488\,\#1^{38}+1\,410\,471\,953\,408\,\#1^{40}-443\,259\,328\,512\,\#1^{42}+83\,609\,604\,096\,\#1^{44}-10\,181\,738\,496\,\#1^{46}+715\,784\,192\,\#1^{48}-30\,081\,024\,\#1^{50}+131\,072\,\#1^{52}\&,\,7\right]\right]\}\}$$

...namely, $\tilde{t}_0 = 2\arctan\tilde{\alpha}$, where $\tilde{\alpha}$ is the seventh root of the polynomial

```
131 072 - 30 081 024 #1^2 + 715 784 192 #1^4 - 10 181 738 496 #1^6 + 83 609 604 096 #1^8 -
    443 259 328 512 #1^10 + 1 410 471 953 408 #1^12 - 1 858 643 071 488 #1^14 +
    18 137 673 285 920 #1^16 - 14 367 112 128 688 #1^18 + 56 162 265 469 488 #1^20 -
    73 041 229 883 512 #1^22 + 73 382 345 772 378 #1^24 - 122 601 623 733 111 #1^26 +
    73 382 345 772 378 #1^28 - 73 041 229 883 512 #1^30 + 56 162 265 469 488 #1^32 -
    14 367 112 128 688 #1^34 + 18 137 673 285 920 #1^36 - 1 858 643 071 488 #1^38 +
    1 410 471 953 408 #1^40 - 443 259 328 512 #1^42 + 83 609 604 096 #1^44 -
    10 181 738 496 #1^46 + 715 784 192 #1^48 - 30 081 024 #1^50 + 131 072 #1^52 &[z]
```

$$131\,072-30\,081\,024\,z^2+715\,784\,192\,z^4-10\,181\,738\,496\,z^6+83\,609\,604\,096\,z^8-443\,259\,328\,512\,z^{10}+1\,410\,471\,953\,408\,z^{12}-1\,858\,643\,071\,488\,z^{14}+18\,137\,673\,285\,920\,z^{16}-14\,367\,112\,128\,688\,z^{18}+56\,162\,265\,469\,488\,z^{20}-73\,041\,229\,883\,512\,z^{22}+73\,382\,345\,772\,378\,z^{24}-122\,601\,623\,733\,111\,z^{26}+73\,382\,345\,772\,378\,z^{28}-73\,041\,229\,883\,512\,z^{30}+56\,162\,265\,469\,488\,z^{32}-14\,367\,112\,128\,688\,z^{34}+18\,137\,673\,285\,920\,z^{36}-1\,858\,643\,071\,488\,z^{38}+1\,410\,471\,953\,408\,z^{40}-443\,259\,328\,512\,z^{42}+83\,609\,604\,096\,z^{44}-10\,181\,738\,496\,z^{46}+715\,784\,192\,z^{48}-30\,081\,024\,z^{50}+131\,072\,z^{52}$$

We verify that the seventh root is indeed the smallest positive root:

```
TableForm@Table[
  {i, N[Root[131 072 - 30 081 024 #1^2 + 715 784 192 #1^4 - 10 181 738 496 #1^6 + 83 609 604 096 #1^8 -
       443 259 328 512 #1^10 + 1 410 471 953 408 #1^12 - 1 858 643 071 488 #1^14 +
       18 137 673 285 920 #1^16 - 14 367 112 128 688 #1^18 + 56 162 265 469 488 #1^20 -
       73 041 229 883 512 #1^22 + 73 382 345 772 378 #1^24 - 122 601 623 733 111 #1^26 +
       73 382 345 772 378 #1^28 - 73 041 229 883 512 #1^30 + 56 162 265 469 488 #1^32 -
       14 367 112 128 688 #1^34 + 18 137 673 285 920 #1^36 - 1 858 643 071 488 #1^38 +
       1 410 471 953 408 #1^40 - 443 259 328 512 #1^42 + 83 609 604 096 #1^44 - 10 181 738 496 #1^46 +
       715 784 192 #1^48 - 30 081 024 #1^50 + 131 072 #1^52 &, i]]}, {i, 1, 52}]
```

1	−14.3035
2	−2.84231
3	−1.04403
4	−0.957824
5	−0.351827
6	−0.0699128
7	0.0699128
8	0.351827
9	0.957824
10	1.04403
11	2.84231
12	14.3035
13	−2.90065 − 1.45724 i
14	−2.90065 + 1.45724 i
15	−2.40338 − 1.59391 i
16	−2.40338 + 1.59391 i
17	−0.815224 − 1.4638 i
18	−0.815224 + 1.4638 i
19	−0.813385 − 0.925832 i
20	−0.813385 + 0.925832 i
21	−0.664394 − 0.787029 i
22	−0.664394 + 0.787029 i
23	−0.626294 − 0.741897 i
24	−0.626294 + 0.741897 i
25	−0.535559 − 0.609597 i
26	−0.535559 + 0.609597 i
27	−0.290393 − 0.521426 i
28	−0.290393 + 0.521426 i
29	−0.28898 − 0.19165 i
30	−0.28898 + 0.19165 i
31	−0.275274 − 0.138294 i
32	−0.275274 + 0.138294 i
33	0.275274 − 0.138294 i
34	0.275274 + 0.138294 i
35	0.28898 − 0.19165 i
36	0.28898 + 0.19165 i
37	0.290393 − 0.521426 i
38	0.290393 + 0.521426 i
39	0.535559 − 0.609597 i
40	0.535559 + 0.609597 i
41	0.626294 − 0.741897 i
42	0.626294 + 0.741897 i
43	0.664394 − 0.787029 i
44	0.664394 + 0.787029 i
45	0.813385 − 0.925832 i
46	0.813385 + 0.925832 i
47	0.815224 − 1.4638 i
48	0.815224 + 1.4638 i
49	2.40338 − 1.59391 i
50	2.40338 + 1.59391 i
51	2.90065 − 1.45724 i
52	2.90065 + 1.45724 i

And the decimal approximation of $\tilde{\alpha}$ is

```
N[Root[131 072 - 30 081 024 #1^2 + 715 784 192 #1^4 - 10 181 738 496 #1^6 + 83 609 604 096 #1^8 -
    443 259 328 512 #1^10 + 1 410 471 953 408 #1^12 - 1 858 643 071 488 #1^14 +
    18 137 673 285 920 #1^16 - 14 367 112 128 688 #1^18 + 56 162 265 469 488 #1^20 -
    73 041 229 883 512 #1^22 + 73 382 345 772 378 #1^24 - 122 601 623 733 111 #1^26 +
    73 382 345 772 378 #1^28 - 73 041 229 883 512 #1^30 + 56 162 265 469 488 #1^32 -
    14 367 112 128 688 #1^34 + 18 137 673 285 920 #1^36 - 1 858 643 071 488 #1^38 +
    1 410 471 953 408 #1^40 - 443 259 328 512 #1^42 + 83 609 604 096 #1^44 -
    10 181 738 496 #1^46 + 715 784 192 #1^48 - 30 081 024 #1^50 + 131 072 #1^52 &, 7]]
0.0699128
```