ALGEBRA HW 8

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1

Find all automorphisms of the ring $\mathbb{Z}[x]$.

Answer: First, we note that, if $\phi : \mathbb{Z}[x] \to \mathbb{Z}[x]$ is an automorphism, then $\phi(1) = 1$, which in turn means that $\phi(c) = c$ for all constant terms c. Hence, ϕ is completely determined by $\phi(x)$. Suppose

$$\deg(\phi(x)) = d.$$

Then, for all non-constant polynomials $f(x) \in \mathbb{Z}[x]$,

$$\deg(\phi(f(x))) \geq d$$

since $f(x)$ is a linear combination of multiples of x. However, since ϕ is an automorphism, it must be surjective, so there exists non-constant $f(x) \in \mathbb{Z}[x]$ such that $\phi(f(x)) = x$. Hence, it must be the case that $d = 1$, which is to say that

$$\phi(x) = \alpha x.$$

Suppose $\alpha = pq$ for $p, q \in \mathbb{Z}$, and $\phi(g(x)) = qx$. Then

$$\phi(x) = \alpha x = pqx = \phi(p)\phi(g(x)) = \phi(pg(x)).$$

However, the only values of p and g for which this could be satisfied are $p = \pm 1$, $g(x) = \pm x$. Hence, there are only two automorphisms of $\mathbb{Z}[x]$, corresponding to $\phi_1 : \mathbb{Z}[x] \to \mathbb{Z}[x]$ and $\phi_2 : \mathbb{Z}[x] \to \mathbb{Z}[x]$, where $\phi_1(x) = x$ and $\phi_2(x) = -x$.

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2

Determine all maximal ideals of the ring $\mathbb{Z}[\frac{1}{2}]$, and show that each maximal ideal can be generated by one element.

Answer:

3

Let m be the ideal of $\mathbb{Z}[x]$ generated by 5 and x. Prove that m is a maximal ideal.

Proof. Let $\phi : \mathbb{Z}[x] \to \mathbb{F}_5$ be given by

$$\sum_{i=0}^{\infty} a_i x^i \mapsto [a_0]$$

where $[a_0] = a_0 + 5\mathbb{Z} \in \mathbb{F}_5$. Then, if $\sum a_i x^i, \sum b_i x^i \in \mathbb{Z}[x]$,

$$\phi\left(\sum a_i x^i + b_i x^i\right) = \phi\left(\sum (a_i + b_i) x^i\right) = a_0 + b_0 = \phi\left(\sum a_i x^i\right) + \phi\left(\sum b_i x^i\right)$$

and

$$\phi\left(\left(\sum a_i x^i\right)\left(\sum b_j x^j\right)\right) = \phi\left(\sum_{i+j=k} a_i b_j x^k\right) = a_0 b_0 = \phi\left(\sum a_i x^i\right)\phi\left(\sum b_j x^j\right),$$

so ϕ is a ring homomorphism. Hence, by the First Isomorphism Theorem,

$$\mathbb{Z}[x]/\ker(\phi) \simeq \operatorname{Im}(\phi) = \mathbb{F}_5$$

since ϕ is surjective, so $\ker(\phi)$ is a maximal ideal of $\mathbb{Z}[x]$. Now,

$$\ker(\phi) = \left\{\sum a_i x^i : a_i = 5x \text{ for some } x \in \mathbb{Z}\right\} = m$$

the ideal of $\mathbb{Z}[x]$ generated by 5 and x. Therefore, since $\mathbb{F}_5$ is a field, m is maximal. $\square$

4

Let $\mathbb{Q}[\sqrt{-1}]$ be the subset of $\mathbb{C}$ consisting of all numbers of the form $a + b\sqrt{-1}$, $a, b \in \mathbb{Q}$. The group $\mathbb{Z}/4\mathbb{Z}$ operates on $\mathbb{Q}[\sqrt{-1}]$, such that the element $n + 4\mathbb{Z}$ acts as "multiplication by $\sqrt{-1}^n$" for any $n \in \mathbb{Z}$. The above action induces a ring homomorphism $\alpha : \mathbb{Q}[\mathbb{Z}/4\mathbb{Z}] \to \operatorname{End}_{\mathbb{Q}}(\mathbb{Q}[\sqrt{-1}])$, where $\operatorname{End}_{\mathbb{Q}}(\mathbb{Q}[\sqrt{-1}])$ denotes the ring of all $\mathbb{Q}$-linear endomorphisms of the $\mathbb{Q}$-vector space $\mathbb{Q}[\sqrt{-1}]$. Prove that α is *not* injective.

Proof. Let $[n] := n + 4\mathbb{Z}$. Then $\alpha : \mathbb{Q}[\mathbb{Z}/4\mathbb{Z}] \to \operatorname{End}_{\mathbb{Q}}(\mathbb{Q}[\sqrt{-1}])$ is given by

$$[n] \mapsto (x \mapsto x\sqrt{-1}^n).$$

Then

$$\alpha([1]) = (x \mapsto x\sqrt{-1}) = (x \mapsto -x(-\sqrt{-1})) = (x \mapsto -x\sqrt{-1}^3) = \alpha(-[3]).$$

Hence, α is not injective. $\square$

5

Prove that $\mathbb{R}[x]$ is *not* finite dimensional over $\mathbb{R}$.

Proof. By contradiction. Suppose $\mathbb{R}[x]$ were finite dimensional over $\mathbb{R}$. Then there exists some finite basis $B = \{b_1, \ldots, b_n\}$ for $\mathbb{R}[x]$ over $\mathbb{R}$. Certainly, $b_i \neq 0$ for all $i = 1, \ldots n$, so $\deg(b_i)$ is well-defined. Let

$$m = \max_{i \in \{1,\ldots,n\}} (\deg(b_i)).$$

Then $x^{m+1} \in \mathbb{R}[x]$, but there is no way to write x^{m+1} as an $\mathbb{R}$-linear combination of the b_i's. From this contradiction, then, we conclude that, in fact, $\mathbb{R}[x]$ is not finite dimensional over $\mathbb{R}$. $\square$

6

The three elements of $\mathbb{Z}/3\mathbb{Z}$ gives an $\mathbb{R}$-basis of the group algebra $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$. For each element $n + 3\mathbb{Z}$ of the group $\mathbb{Z}/3\mathbb{Z}$, write down the matrix representation of $n + 3\mathbb{Z}$ acting on the 3-dimensional vector space $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$.

Answer: We consider each $h_n : \mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \to \mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$ given by

$$x \mapsto (n + 3\mathbb{Z}) + x.$$

Then,

$$h_0(0 + 3\mathbb{Z}) = 0 + 3\mathbb{Z}, \quad h_0(1 + 3\mathbb{Z}) = 1 + 3\mathbb{Z}, \quad h_0(2 + 3\mathbb{Z}) = 2 + 3\mathbb{Z},$$

so the matrix representation is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now,

$$h_1(0 + 3\mathbb{Z}) = 1 + 3\mathbb{Z}, \quad h_1(1 + 3\mathbb{Z}) = 2 + 3\mathbb{Z}, \quad h_1(2 + 3\mathbb{Z}) = 0 + 3\mathbb{Z},$$

so the matrix representation is:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Finally,

$$h_2(0 + 3\mathbb{Z}) = 2 + 3\mathbb{Z}, \quad h_2(1 + 3\mathbb{Z}) = 0 + 3\mathbb{Z}, \quad h_2(2 + 3\mathbb{Z}) = 1 + 3\mathbb{Z},$$

so the matrix representation is:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

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7

Find a maximal ideal of the group algebra $\mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}]$.

Answer: Denote by $[n]$ the element $n + 3\mathbb{Z}$. Let $\phi : \mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}] \to \mathbb{F}_7$ be given by

$$a_0[0] + a_1[1] + a_2[2] \mapsto a_0 + a_1 + a_2,$$

where $a_0, a_1, a_2 \in \mathbb{F}_7$. This map is certainly surjective, as, for any $a \in \mathbb{F}_7$,

$$\phi : a[0] + 0[1] + 0[2] \mapsto a + 0 + 0 = a.$$

Also, for $a_0[0] + a_1[1] + a_2[2], b_0[0] + b_1[1] + b_2[2] \in \mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}]$,

$$\begin{aligned}
&\phi((a_0[0] + a_1[1] + a_2[2]) + (b_0[0] + b_1[1] + b_2[2])) \\
&= (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) \\
&= (a_0 + a_1 + a_2) + (b_0 + b_1 + b_2) \\
&= \phi(a_0[0] + a_1[1] + a_2[2]) + \phi(b_0[0] + b_1[1] + b_2[2])
\end{aligned}$$

and

$$\begin{aligned}
&\phi((a_0[0] + a_1[1] + a_2[2])(b_0[0] + b_1[1] + b_2[2])) \\
&= \phi((a_0b_0 + a_1b_2 + a_2b_1)[0] + (a_0b_1 + a_1b_0 + a_2b_2)[1] + (a_0b_2 + a_2b_2 + a_1b_1)[2]) \\
&= (a_0b_0 + a_1b_2 + a_2b_1) + (a_0b_1 + a_1b_0 + a_2b_2) + (a_0b_2 + a_2b_0 + a_1b_1) \\
&= (a_0 + a_1 + a_2)(b_0 + b_1 + b_2) \\
&= \phi(a_0[0] + a_1[1] + a_2[2])\phi(b_0[0] + b_1[1] + b_2[2])
\end{aligned}$$

so ϕ is a homomorphism. Hence, by the First Isomorphism Theorem,

$$\mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}]/\ker(\phi) \simeq \mathbb{F}_7$$

since ϕ is surjective. Since $\mathbb{F}_7$ is a field, $\ker(\phi)$ is a maximal ideal of $\mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}]$.

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